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On weakly β-γ-regular and weakly β-γ-normal spaces

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Abstract: In this paper, we introduce, investigate and study two new topological spaces namely weakly β-γ-regular and weakly β-γ-normal spaces, by using the concept of operational approach on β-open sets. The concept of β-γ-open sets was introduced, and also we utilized β-γ-open sets on β-regular and β-normal spaces to study and investigate these new spaces. Some topological properties of weakly β-γ-regular and weakly β-γ-normal spaces were also studied. Some hereditary properties and preservation theorems are also being obtained.

Key words: operation, β-γ-open set, weakly β-γ-regular, weakly β-γ-normal

1. Introduction and Preliminaries

Topology is the branch of mathematics, whose concepts exist not only in all branches of mathematics, but also in diverse field outside the realm of mathematics. Several researchers are working on different structures of topological spaces. That give rise to new spaces and new topological sets and concepts, even it had generalized the notation of fuzzy sets too. Ogata [4], and Jankovic [3] defined operational approach on sets and researched new operational approach on sets and obtained its basic properties.We used this notation to study new spaces namely weakly β-γ-regular and weakly β-γ-normal spaces by using the concept of operational approach on β-open sets. An operation $\gamma : \beta O(X) \to P(X)$ is a mapping satisfying the condition $V \subseteq V^{\gamma}$ for each $V \in \beta O(X)$ and $\delta : \beta O(Y) \to P(Y)$ is a mapping satisfying the condition $V \subseteq V^{\delta}$ for each $V \in \beta O(Y)$. The operation $id : \beta O(X) \to P(X)$ defined by $V^{id} = V$ for any set $V \in \beta O(X)$ is called the identity operation on $\beta O(X)$ [5].

Throughout the paper, (X, τ) and (Y, σ) represent non empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For any subset A of X, $Cl(A)$, $Int(A)$ denote the closure, and the interior of A, respectively. A subset A of a topological space X is said to be β-open [1] if $A \subseteq Cl(Int(Cl(A)))$. The collection of all β-open sets is denoted by $\beta O(X)$ and the complement of a β-open set is β-closed [1]. The intersection of all β-closed sets containing A is called β-closure of A and is denoted by $\beta Cl(A)$ [2] and the union of all β-open sets contained in A is called as β-interior of A and is denoted by $\beta Int(A)$ [2].

2. Definitions

Definition 2.1 ([5]). (i) A subset A of X is called β-γ-open set if for each point $x \in A$, there exists a β-open set U of X containing x such that $U^{\gamma} \subseteq A$. The complement of β-γ-open set is called β-γ-closed. The set of all β-γ-open sets of (X, τ) are denoted by $\beta O(X, \tau)_{\gamma}$.

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(ii) The β-γ-closure of a subset A of X, with an operation γ on $\beta O(X)$, denoted by $\beta_\gamma Cl(A)$, is defined to be the intersection of all β-γ-closed sets containing A. The β-γ-interior of A is the union of all β-γ-open set contained in A and it is denoted by $\beta_\gamma Int(A)$. The β-γ-boundary of a set A is defined as $(\beta_\gamma Cl(A)$-$\beta_\gamma Int(A))$ and is denoted by $\beta_\gamma Bd(A)$.

(iii) A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be

(a) $\beta_{(\gamma,\delta)}$-continuous if for each x of X and each β-δ-open set V containing $f(x)$, there exists a β-γ-open set U such that $x\in U$ and $f(U)\subseteq V$.

(b) $\beta_{(\gamma,\delta)}$-closed if for each β-γ-closed set A of (X,τ), the set $f(A)$ is β-δ-closed in (Y,σ).

(c) $\beta_{(\gamma,\delta)}$-open if for each β-γ-open set A of (X,τ), the set $f(A)$ is β-δ-open in (Y,σ).

(d) β-(γ,b)-continuous if for each $x\in X$ and each β-open set V containing $f(x)$, there exists a β-open set U such that $x\in U$ and $f(U^\gamma)\subseteq V^b$.

(iv) A space (X,τ) is said to be

(a) β-γ-regular if for each $x\in X$ and for each β-open set V containing x, there exists a β-open set U containing x such that $U^\gamma\subseteq V$.

(b) β-γ-T_1 if for any two distinct points $x,y\in X$, there exists two β-open sets U and V containing x, y respectively such that $y\notin U^\gamma$ and $x\notin V^\gamma$.

(c) β-γ-T_1' if for any two distinct points $x,y\in X$, there exist two β-γ-open sets U and containing x, y respectively such that $y\notin U$ and $x\notin V$.

(d) Let γ be an operation on $\beta O(X)$. Then γ is said to be β-regular if for each $x\in X$ and for every pair of β-open sets U and V containing x, there exists a β-open set W such that $x\in W$ and $W^\gamma\subseteq U^\gamma\cap V^\gamma$.

(e) A subset A of a space X is said to be β-γ-generalized closed (briefly β-γ-g.closed) if $\beta_\gamma Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is β-γ-open set in X. The complement of β-γ-g.closed set is β-γ-g.open.

Theorem 2.1.

(i) *For a topological space (X,τ) and an operation γ on $\beta O(X)$. The following properties hold [5]:*

(a) (X,τ) *is* β-γ-T_1.

(b) *For every point $x\in X$, $\{x\}$ is a β-γ-closed set.*

(c) (X,τ) *is* β-γ-T_1'.

(ii) *The following statements are equivalent*

(a) $\beta O(X,\tau)=\beta O(X,\tau)_\gamma$.

(b) (X,τ) *is* β-γ-*regular space.*

(c) *For every* $x \in X$ *and every* β*-open set* U *of* (X,τ) *containing* x*, there exists a* β*-*γ*-open set* W *of* (X,τ) *such that* $x \in W$ *and* $W \subseteq V$*.*

Definition 2.2. Let γ be an operation on $\beta O(X)$. Then γ is said to β-monotone if for all $A, B \in \beta O(X)$ and $A \subseteq B$, then $A^\gamma \subseteq B^\gamma$.

Definition 2.3. A point $x \in X$ is said to be β-γ-limit point of the set A if for each β-γ-open U containing x, then $U \cap (A - \{x\}) \neq$. The set of all β-γ-limit points of A is called a β-γ-derived set of A and is denoted by $\beta_\gamma D(A)$.

Theorem 2.2. *Let* A *be any subset of a topological space* (X,τ) *and* γ *be an operation on* $\beta O(X)$*. Then*

(i) $\beta_\gamma Int(A) = A - \beta_\gamma D(X - A)$.

(ii) X-$\beta_\gamma Int(A) = \beta_\gamma Cl(X - A)$.

(iii) $\beta_\gamma Int(A) = X - \beta_\gamma Cl(X - A)$.

(iv) $\beta_\gamma Cl(A) = X - \beta_\gamma Int(X - A)$.

Proof. (i): If $x \in A - \beta_\gamma D(X - A)$, then $x \notin \beta_\gamma D(X - A)$ and so there exists a β-γ-open set U containing x such that $U \cap (X - A) =$. Then $x \in U \subseteq A$ and hence $x \in \beta_\gamma Int(A)$, so A-$\beta_\gamma D(X - A) \subseteq \beta_\gamma Int(A)$. Also if $x \in \beta_\gamma Int(A)$, then $x \notin \beta_\gamma D(X - A)$ since $\beta_\gamma Int(A)$ is β-γ-open and $\beta_\gamma Int(A) \cap (X - A) =$. Hence $\beta_\gamma Int(A) = A - \beta_\gamma D(X - A)$.

(ii): $X - \beta_\gamma Int(A) = X - (A - \beta_\gamma D(X - A)) = (X - A) \cup \beta_\gamma D(X - A) = \beta_\gamma Cl(X - A)$.

(iii), (iv). Similar. □

Theorem 2.3. *Let* γ *be a* β*-monotone on* $\beta O(X)$*. If* A *is a subset of* X*, then*

1. *For every* β*-*γ*-open set* G *of* X*, we have* $\beta Cl_\gamma(A) \cap G \subseteq \beta Cl_\gamma(A \cap G)$*.*
2. *For every* β*-*γ*-closed set* F *of* X*, we have* $\beta Int_\gamma(A \cup F) \subseteq \beta Int_\gamma(A) \cup F$*.*

Proof. 1. Let $x \in \beta Cl_\gamma(A) \cap G$ and let U be a β-open set containing x. Since $x \in \beta Cl_\gamma(A)$, implies that $U^\gamma \cap A \neq$. Since G is a β-γ-open set, there exists a β-open set V containing x such that $V^\gamma \subseteq G$. Thus $(U \cap V)^\gamma \cap A \neq$ (as $U \cap V \subseteq U$ and $U \cap V \subseteq U$) and this implies that $U^\gamma \cap (A \cap G) \neq$ as γ is β-monotone and hence $x \in \beta Cl_\gamma(A \cap G)$. So $\beta Cl_\gamma(A) \cap G \subseteq \beta Cl_\gamma(A \cap G)$.

2. Similar to 1. □

Theorem 2.4. *A subset* A *of* X *is* β_γ*-g.open if and only if* $F \subseteq \beta_\gamma Int(A)$ *whenever* $F \subseteq AandF$ *is* β*-*γ*-closed set in* X*.*

Proof. Let A be β_γ-g.open and $F \subseteq A$, where F is β-γ-closed. Since $X - A$ is β_γ-g.closed and $X - F$ is β-γ-open set containing $X - A$ implies $\beta_\gamma Cl(X - A) \subseteq X - F$. By Theorem 2.2(i), $X - \beta_\gamma Int(A) \subseteq X - F$. So $F \subseteq \beta_\gamma Int(A)$.

Conversely, suppose that F is β-γ-closed and $F \subseteq A$ implies $F \subseteq \beta_\gamma Int(A)$. Let $X - A \subseteq U$, where U is β-γ-open. Then $X - U \subseteq A$ where $X - U$ is β-γ-closed. By hypothesis, $X - U \subseteq \beta_\gamma Int(A)$. So

$X - \beta_\gamma Int(A) \subseteq U$. By Theorem 2.2(i), $\beta_\gamma Cl(X - A) \subseteq U$. This implies $X - A$ is β_γ-g.closed and hence A is β_γ-g.open. □

3. Weakly β-γ-regular and Weakly β-γ-normal Spaces

Definition 3.1. A space X is said to be weakly β-γ-regular, if for any β-γ-closed set A and $x \notin A$, there exist disjoint β-γ-open sets U and V such that $x \in U$, $A \subseteq V$.

Following are the examples which implies that weakly β-γ-regular spaces are independent of regular spaces and conversely:

Example 3.1. (i) *Consider $X = \{a, b, c\}$ with discrete topology τ on X. For a non empty set A, we define an operation γ on $\beta O(X)$ by*

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{b, c\} \\ X & \text{if otherwise} \end{cases}$$

Then, the space X is regular but not weakly β-γ-regular.

(ii) *Consider $X = \{a, b, c\}$ with topology $\tau = \{, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. For a non empty set A, we define an operation γ on $\beta O(X)$ by*

$$A^\gamma = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{b, c\} \\ X & \text{if otherwise.} \end{cases}$$

Then X is β-γ-regular but not regular.

We now give some results on weakly β-γ-regular spaces:

Theorem 3.1. *Consider the following statements on a space X.*

1. *X is weakly β-γ-regular,*
2. *For any β-γ-open set U in X and $x \in U$, there is a β-γ-open set V containing x such that $\beta Cl_\gamma(V) \subseteq U$.*
3. *For every point $x \in X$ and every β-γ-nbhd N of x, there exists a β-closed set B such that $x \in B \subseteq N$.*

Proof. (1)⇒(2). Let U be a β-γ-open set and $x \in U$. Then $X - U$ is a β-γ-closed set such that $x \notin X - U$. By weak β-γ-regularity of X, there are β-γ-open sets V and G such that $x \in V$, $X - U \subseteq G$ and $V \cap G = \emptyset$. Clearly, $X - G$ is a β-γ-closed set contained in U. Then $V \subseteq X - G \subseteq U$. This gives that $\beta Cl_\gamma(V) \subseteq X - G \subseteq U$. Consequently, $x \in V$ and $\beta Cl_\gamma(V) \subseteq U$.

(2)⇒(3). Let $x \in X$ be any element and N be any β-γ-nbhd of x. By definition, there exists a β-γ-open set U such that $x \in U \subseteq N$. By (2), there is a β-γ-open set V containing x such that $\beta Cl_\gamma(V) \subseteq U$. Then $x \in \beta Cl_\gamma(V) \subseteq N$ and since $\beta Cl_\gamma(V)$ is β-closed set by. Thus for each $x \in X$, the set N forms a β-γ-neighbourhood consisting of β-closed set of X. This proves (3).

In general, (3)⇒(1) does not hold as can be seen from Example 3.1(i). □

Theorem 3.2. *The following properties are equivalent for a topological space (X, τ) with an operation γ on $\beta O(X, \tau)$.*

(1) *X is weakly β-γ-regular.*

(2) *For each $x \in X$ and each β-γ-open set U containing x, there exists a β-γ-open set V such that $x \in V \subseteq \beta_\gamma Cl(V) \subseteq U$.*

(3) *For each β-γ-closed subset F of X, $F = \cap\{\beta_\gamma Cl(V) : F \subseteq V and V \in \beta O(X,\tau)_\gamma\}$.*

(4) *For each subset A of X and each $U \in \beta O(X,\tau)$ with $A \cap U \neq$, there exists $V \in \beta O(X,\tau)$ such that $A \cap V \neq$ and $\beta_\gamma Cl(V) \subseteq U$.*

(5) *For each non empty subset A of X and each β-γ-closed set F of X such that $A \cap F =$, there exist $V, W \in \beta O(X,\tau)_\gamma$ such that $A \cap V \neq$, $F \subseteq W$ and $W \cap V =$.*

(6) *For each β-γ-closed set F and $x \notin F$, there exists $U \in \beta O(X,\tau)_\gamma$ and an β_γ-g.open set V such that $x \in U$, $F \subseteq V$ and $U \cap V =$.*

(7) *For each $A \subseteq X$ and each β-γ-closed set F with $A \cap F =$, there exists $U \in \beta O(X,\tau)_\gamma$ and a β_γ-g.open set V such that $A \cap U \neq$, $F \subseteq V$ and $U \cap V =$.*

(8) *For each β-γ-closed set F of $X, F = \cap\{\beta_\gamma Cl(V) : F \subseteq V$ and V is a β_γ-g.open$\}$.*

Proof. (1)⇒(2). Let $x \notin X - U$, where U is a β-γ-open set containing x. Then by (1), there exists $G, V \in \beta O(X,\tau)_\gamma$ such that $X - U \subseteq G$, $x \in V$ and $G \cap V =$. Therefore $V \subseteq X - G$ and so $x \in V \subseteq \beta_\gamma Cl(V) \subseteq X - G \subseteq U$.

(2)⇒(3). Let $X - F$ be any $\beta - \gamma$-open set containing x. Then by (2), there exists a $\beta - \gamma$-open set U containing x such that $x \in U \subseteq \beta_\gamma Cl(U) \subseteq X - F$. So $F \subseteq X - \beta_\gamma Cl(U) = V$, $V \in \beta O(X,\tau)_\gamma$ and $V \cap U =$. Then by, $x \notin \beta_\gamma Cl(V)$. Hence, we obtain that $F \supseteq \cap\{\beta_\gamma Cl(V) : F \subseteq V$ and $V \in \beta O(X,\tau)_\gamma\}$.

(3)⇒(4). Let $U \in \beta O(X,\tau)_\gamma$ with $x \in U \cap A$. Then $x \notin X - U$ and hence by (3), there exists a $\beta - \gamma$-open set W such that $X - U \subseteq W$ and $x \notin \beta_\gamma Cl(W)$. We put $V = X - \beta_\gamma Cl(W)$, which is a $\beta - \gamma$-open set containing x and hence $V \cap A \neq$. Now $V \subseteq X - W$ and so $\beta_\gamma Cl(V) \subseteq X - W \subseteq U$.

(4)⇒(5). Let F be a $\beta - \gamma$-closed set.Then $X - F$ is a $\beta - \gamma$-open set and $(X - F) \cap A \neq$. Then, there exists $V \in \beta O(X,\tau)_\gamma$ such that $A \cap V \neq$ and $\beta_\gamma Cl(V) \subseteq X - F$. If we put $W = X - \beta_\gamma Cl(V)$, then $F \subseteq W$ and $W \cap V =$.

(5)⇒(1). Let F be any β-γ-closed set not containing x. Then, there exists $W, V \in \beta O(X,\tau)_\gamma$ such that $F \subseteq W$ and $x \in V$ and $W \cap V =$.

(1)⇒(6). Clear.

(6)⇒(7). For $a \in A$, $a \notin F$ and hence by (6), there exists $U \in \beta O(X,\tau)_\gamma$ and a β_γ-g.open set V such that $a \in U$, $F \subseteq V$ and $U \cap V =$. So, $A \cap U \neq$.

(7)⇒(1). Let $x \notin F$, where F is β-γ-closed. Since $\{x\} \cap F =$ by (7), there exists $U \in \beta O(X,\tau)_\gamma$ and an β_γ-g.open set W such that $x \in U$, $F \subseteq W$ and $U \cap W =$. Now put $V = \beta_\gamma Int(W)$. By Theorem 2.4 of β_γ-g.open sets, we get $F \subseteq V$ and $V \cap U =$.

(3)⇒(8). We have $F \subseteq \cap\{\beta_\gamma Cl(V) : F \subseteq V$ and V is a β_γ-g.open$\} \subseteq \cap\{\beta_\gamma Cl(V) : F \subseteq V$ and V is a β_γ-open$\} = F$.

(8)$\Rightarrow$(1). Let F be a β-γ-closed set in X not containing x. Then by (8), there exists a β_γ-g.open set W such that $F \subseteq W$ and $x \in X - \beta_\gamma Cl(V)$. Since F is $\beta - \gamma$-closed and W is β_γ-g.open, then $F \subseteq \beta_\gamma Int(W)$. Let $V = \beta_\gamma Int(W)$. Then $F \subseteq V$, $x \in U = X - \beta_\gamma Cl(V)$ and $U \cap V =$. □

The following theorem shows that weakly β-γ-regularity is a hereditary property.

Theorem 3.3. *Let γ be an operation on $\beta O(X)$ and let $(H, \beta O(X)\gamma|H)$ be a subspace of (X, τ). If X is weakly β-γ-regular, then H is weakly β-γ-regular.*

Proof. Suppose that A is β-γ-closed set in H and $y \in H$ such that $y \notin A$. Then there exists a β-γ-open set U in X such that $H - A = U \cap H$. This implies that $A = B \cap H$, where $B = X - U$ is β-γ-closed set in X. Then $y \notin B$. Since X is weakly β-γ-regular, there exist disjoint β-γ-open sets U and V in X such that $y \in U$, $B \subseteq V$. Then $U \cap H$ and $V \cap H$ are disjoint β-γ-open sets in H containing y and A, respectively. □

Theorem 3.4. *Let (X, τ) be a topological space and let γ be a β-regular operation on $\beta O(X)$. Then X is weakly β-γ-regular operation if and only if for each $x \in X$ and a is β-γ-closed set A such that $x \notin A$, there exists β-γ-open sets U and V in X such that $x \in U$ and $A \subseteq V$ and $\beta_\gamma Cl(U) \cap \beta_\gamma Cl(V) =$.*

Proof. Let $x \in X$ and A be a β-γ-closed set such that $x \notin A$. Then by Theorem 3.1, there is a β-γ-open set W such that $x \in W$, $\beta_\gamma Cl(W) \subseteq X - A$. Again by Theorem 3.1, there exists a β-γ-open set U containing x such that $\beta_\gamma Cl(U) \subseteq W$. Let $V = X - \beta_\gamma Cl(W)$. Then $\beta_\gamma Cl(U) \subseteq W \subseteq \beta_\gamma Cl(W) \subseteq X - A$ implies that $A \subseteq X - \beta_\gamma Cl(W) = V$. Also, $\beta_\gamma Cl(U) \cap \beta_\gamma Cl(V) = \beta_\gamma Cl(U) \cap \beta_\gamma Cl(X - \beta_\gamma Cl(W) \subseteq W \cap \beta_\gamma Cl(X - \beta_\gamma Cl(W)) \subseteq \beta_\gamma Cl(W \cap X - \beta_\gamma Cl(W)) = \beta_\gamma Cl() =$ (by Theorem 2.3). Thus U and V are required β-γ-open sets in X. This proves the necessity. Converse is obvious. □

Definition 3.2. A space X is said to be weakly β-γ-normal, if for any disjoint a β-γ-closed sets A and B of X, there exist β-γ-open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V =$.

Following are the examples which implies that weakly β-γ-normal spaces are independent of normal spaces and conversely:

Example 3.2.

(i) *Consider $X = \{a, b, c\}$ with topology $\tau = \{, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ on X. For a non empty set A, we define an operation γ on $\beta O(X)$ by*

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{if otherwise} \end{cases}$$

Then the space X is normal but not weakly β-γ-normal.

(ii) *Consider $X = \{a, b, c\}$ with topology $\tau = \{, X, \{a\}, \{a, b\}, \{a, c\}\}$ on X. For a non empty set A, we define an operation γ on $\beta O(X)$ by*

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \\ X & \text{if otherwise.} \end{cases}$$

Then X is not normal but it is weakly β-γ-normal.

Next, we give some characterizations of weakly β-γ-normal spaces.

Theorem 3.5. *Let (X,τ) be a topological space and let γ be a operation on $\beta O(X)$. Then X is weakly β-γ-normal if and only if for any β-γ-closed set A and a β-γ-open set U containing A, there is a β-γ-open set V such that $A \subseteq V \subseteq \beta_\gamma Cl(V) \subseteq U$.*

Proof. Since U is a β-γ-open set containing A, then $X-U$ is β-γ-closed and $A \cap (X-U) = $. Since X is weakly β-γ-normal, there exists β-γ-open sets V and V_1 such that $A \subseteq V$, $X-U \subseteq V_1$ and $V \cap V_1 = $. Hence, $A \subseteq V \subseteq \beta_\gamma Cl(V) \subseteq \beta_\gamma Cl(X-V_1) = X - V_1 \subseteq U$ or $A \subseteq V \subseteq \beta_\gamma Cl(V) \subseteq U$.

Consequently, let A and B be the two disjoint β-γ-closed sets in X. Then $A \subseteq X-B$ where $X-B$ is β-γ-open in X. By hypothesis, there is a β-γ-open set V such that $A \subseteq V \subseteq \beta_\gamma Cl(V) \subseteq X-B$, implies that $B \subseteq X - \beta_\gamma Cl(V)$ and $V \cap (X - \beta_\gamma Cl(V)) = $. Consequently, $A \subseteq V$, $B \subseteq X - \beta_\gamma Cl(V)$. This proves that X is weakly β-γ-normal. □

Theorem 3.6. *For a topological space (X,τ) with an operation γ on $\beta O(X,\tau)$, the following statements are equivalent:*

(1) *X is weakly β-γ-normal.*

(2) *For each pair of disjoint β-γ-closed sets A, B of X, there exist disjoint β_γ-g.open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

(3) *For each β-γ-closed set A and any β-γ-open set V containing A, there exists an β_γ-g.open set U such that $A \subseteq U \subseteq \beta_\gamma Cl(U) \subseteq V$.*

(4) *For each β-γ-closed set A and any β_γ-g.open set B containing A, there exists a β_γ-g.open set U such that $A \subseteq U \subseteq \beta Cl(U) \subseteq \beta_\gamma Int(B)$.*

(5) *For each β-γ-closed set A and any β_γ-g.open set B containing A, there exists a β-γ-open set G such that $A \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq \beta_\gamma Int(B)$.*

(6) *For each β_γ-g.closed set A and any β-γ-open set B containing A, there exists a β-γ-open set U such that $\beta_\gamma Cl(A) \subseteq U \subseteq \beta_\gamma Cl(U) \subseteq B$.*

(7) *For each β_γ-g.closed set A and any β-γ-open set B containing A, there exists a β_γ-g.open set G such that $\beta_\gamma Cl(A) \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq B$.*

Proof. (1)⇒(2). Follows from the fact that every β-γ-open set is β_γ-g.open.

(2)⇒(3). Let A be a β-γ-closed subset and let V be an open β-γ-open set containing A. Since A and $X \backslash V$ are disjoint β-γ-closed subsets of X, there exist β_γ-g.open sets U and W of X such that $A \subseteq U$ and $X \backslash V \subseteq W$ and $U \cap W = $. By Theorem 2.4, we get $X \backslash V \subseteq \beta_\gamma Int(W)$. Since $U \cap \beta_\gamma Int(W) = $, we have $\beta_\gamma Cl(U) \cap \beta_\gamma Int(W) = $ and hence $\beta_\gamma Cl(U) \subseteq X \backslash \beta_\gamma Int(W) \subseteq V$. Therefore, we obtain $A \subseteq U \subseteq \beta_\gamma Cl(U) \subseteq V$.

(3)⇒(1). Let A and B be the disjoint β-γ-closed subsets of X. Since $X \backslash B$ is an β-γ-open set containing A, there exists a β_γ-g.open set G such that $A \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq X \backslash B$. By Theorem 2.4, we have $A \subseteq \beta_\gamma Int(G)$. Put $U = \beta_\gamma Int(G)$ and $V = X \backslash \beta_\gamma Cl(G)$. Then U and V are disjoint β-γ-open sets such that $A \subseteq U$ and $B \subseteq V$. Therefore X is weakly β-γ-normal.

It is obvious that (5)⇒(4) (4)⇒(3) and (6)⇒(7)⇒(3).

(5)⇒(3). Let A be any β-γ-closed in X and let B be a β-γ-open set such that $A \subseteq B$. Since every β-γ-open set is β_γ-g.open, there exist a β-γ-open set G such that $A \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq \beta_\gamma Int(B)$. Hence we have $A \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq \beta_\gamma Int(B) \subseteq B$.

(3)⇒(5). Let A be β-γ-closed in X and let B be a β_γ-g.open set such that $A \subseteq B$. Using Theorem 2.4, there exist β_γ-g.open set such that $A \subseteq \beta_\gamma Int(B) = V$, say. Then applying (3), we get β_γ-g.open set set U such that $A = \beta_\gamma Cl(A) \subseteq U \subseteq \beta_\gamma Cl(U) \subseteq V$. Again, using the same proposition, we get $A \subseteq \beta_\gamma Int(U)$ and hence put $U = \beta Int(G)$, then U is β-open and $A \subseteq \beta_\gamma Int(U) \subseteq U \subseteq \beta_\gamma Cl(V) \subseteq V$, which implies $A \subseteq \beta_\gamma Int(U) \subseteq U \subseteq \beta_\gamma Cl(\beta_\gamma Int(U)) \subseteq \beta_\gamma Cl(U) \subseteq V$, that is, $A \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq \beta_\gamma Int(B)$, where $G = \beta_\gamma Int(U)$.

(3)⇒(7). Let A be a β_γ-g-closed set of X and let B be a β-γ-open set such that $A \subseteq B$. Then $\beta_\gamma Cl(A) \subseteq B$ as A is β_γ-g-closed set. Therefore by (3), there exists a β_γ-g.open set U such that $Cl(A) \subseteq U \subseteq \beta_\gamma Cl(U) \subseteq B$.

(7)⇒(6). Let A be a β_γ-g.closed set of X and let B be a β-γ-open set such that $A \subseteq B$. Then there exists a β_γ-g.open set G such that $\beta_\gamma Cl(A) \subseteq G \subseteq \beta_\gamma Cl(G) \subseteq B$. Since G is β_γ-g.open set then by Theorem 2.4, $\beta_\gamma Cl(A) \subseteq \beta_\gamma Int(G)$. If $U = \beta_\gamma Int(G)$. Proof follows. □

Theorem 3.7. *Every weakly β-γ-normal, β-γ-T_1 space is weakly β-γ-regular*

Proof. Suppose that A is β-γ-closed set such that $x \notin A$. Since X is β-γ-T_1 space, then by Theorem 2.1, $\{x\}$ is β-γ-closed in X. Also, since X is weakly β-γ-normal, there exist β-γ-open sets U and V such that $\{x\} \subseteq U$, $A \subseteq V$ and $U \cap V =$ or $x \in U$, $A \subseteq V$ and $U \cap V =$. So X is weakly β-γ-regular. □

Theorem 3.8. *Let γ be an operation on $\beta O(X)$ and $(A, \beta O(X)\gamma|A)$ be a subspace of a topological space (X, τ). If A is β-γ-closed and X is weakly β-γ-normal, then A is weakly β-γ-normal.*

Proof. Let A_1 and A_2 be the disjoint β-γ-closed sets of A. Then there are β-γ-closed sets B_1 and B_2 in X such that $A_1 = B_1 \cap A$ and $A_2 = B_2 \cap A$. Since A is β-γ-closed in X, then A_1 and A_2 are β-γ-closed in X. As X is weakly β-γ-normal, there exists there exist β-γ-open sets U_1 and U_2 such that $A_1 \subseteq U_1$, $A_2 \subseteq U_2$ and $U_1 \cap U_2 =$. But $A_1 \subseteq A \cap U_1$, $A_2 \subseteq A \cap U_2$ where $A \cap U_1$, $A \cap U_2$ are disjoint β-γ-open sets in A. Hence A is weakly β-γ-normal. □

Theorem 3.9. *Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a bijective $\beta_{(\gamma,\delta)}$-continuous and $\beta_{(\gamma,\delta)}$-closed. If X is weakly β-γ-normal, then Y is β-γ-normal.*

Proof. Suppose A_1 and B_1 be the two disjoint β-δ-closed subsets of Y. Then by $\beta_{(\gamma,\delta)}$-continuity of f, $A = f^{-1}(A_1)$, $B = f^{-1}(B_1)$ are disjoint β-γ-closed subsets of X. Since X is weakly β-γ-normal, so for any disjoint a β-γ-closed sets A and B of X, there exist β-γ-open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V =$. Since f is $\beta_{(\gamma,\delta)}$-closed, then $f(X - U)$ and $f(X - V)$ are disjoint β-δ-closed subsets of Y. Then $U_1 = Y - f(X - U)$ and $U_2 = Y - f(X - V)$ are disjoint β-δ-open subsets of Y containing A_1 and B_1, respectively. Hence Y is β-γ-normal. □

Theorem 3.10. *Let γ be a β-regular operation on $\beta O(X)$. Then X is weakly β-γ-normal if and only if for each disjoint β-γ-closed sets A and B of X, there exists β-γ-open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $\beta_\gamma Cl(U) \cap \beta_\gamma Cl(V) =$.*

Proof. The sufficiency is clear. Let A and B be any two disjoint β-γ-closed sets in X. Then $X - B$ is β-γ-open and $A \subseteq X - B$. Then by Theorem 3.5, there is a β-γ-open set C such that $A \subseteq C \subseteq \beta_\gamma Cl(C) \subseteq X - B$. Since $A \subseteq C$, again by Theorem 3.5, there is a a β-γ-open set U such that $\beta_\gamma Cl(U) \subseteq C$. Consequently, $A \subseteq U \subseteq \beta_\gamma Cl(U) \subseteq C$ and $\beta_\gamma Cl(C) \subseteq X - B$ implies that $B \subseteq X - \beta_\gamma Cl(C)$. Put $V = X - \beta_\gamma Cl(C)$. Then V is a β-γ-open set containing B and moreover $\beta_\gamma Cl(U) \cap \beta_\gamma Cl(V) = \beta_\gamma Cl(U) \cap \beta_\gamma Cl(X - \beta_\gamma Cl(C)) \subseteq C \cap \beta_\gamma Cl(X - \beta_\gamma Cl(C)) \subseteq C \cap \beta_\gamma Cl(X - \beta_\gamma Cl(C)) \subseteq \beta_\gamma Cl(C \cap X\text{-}\beta_\gamma Cl(C)) = \beta_\gamma Cl() =$ by Theorem 2.3. Thus U and V are the required β-γ-open sets in X. This proofs the necessity. □

Remark 3.1. *The concept of weakly β-γ-regular and weakly β-γ-normal spaces can further be extended to new topological spaces, as mentioned in [6], [7].*

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