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A discussion of rationality and continuity related to Theta Functions by using the summation of Poisson

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Abstract: Mathematical philosophy is built by defining logical concepts such as the concept of infinity and the concept of continuity. Hence, mathematicians are not those who can calculate, but those who can understand the mathematical concepts and can develop them in order to make new mathematical findings or to transform an hypothesis into a philosophical truth. However, mathematicians are also alarmed scientists who can verify doubted formulas by disproving a mathematical proof or finding a counterexample. It is therefore false to think that mathematicians can't surprise nowadays with new findings.

This is a mathematical demonstration and discussion concerning a special case of functions related to Theta Functions and it is a logical opportunity for those who study the summation of Poisson. This work starts by using a formula of Poisson in order to make a mathematical representation for the studied function then a new approximation of this case of functions is demonstrated. Furthermore, we discuss the consequences if this studied function gives only irrationals as outputs then we use the definition of the continuity of real functions in order to make a demonstration for the continuity of the same Function. Finally, the lovers of mathematics are invited to understand the demonstrations of this article and the discussions of the continuity contradictions in order to develop new findings related to Theta Functions and the summation of Poisson.

Key words: Series, summation, Poisson, Theta function, rational, irrational, limit, logic, approximation, calculus, analysis

1. Introduction

It is a fact that all scientists and engineers need mathematicians. However, many principles of discrete mathematics are not well accepted among people who only apply mathematical tools. I made a previous short note of calculus about the distances on the line of real numbers by considering the example of a runner trying to run in a field which can be integrated by respecting Riemann's definition of the integrals [1]. Here is a new mathematical work that uses the summation of Poisson which is commonly applied in many fields especially for the development of the Zeta Function of Riemann. This work tests if there is a contradiction of continuity that concerns also Riemann Zeta Function and that can be annexed to the previous article that proposes an easy criterion for the study of the zeros of Riemann's hypothesis [2]. This work is dedicated therefore to pure mathematics after my previous works dedicated to mathematical physics like my thesis that proposes new mathematical tools by explaining that physics fields need to apply mathematics differently [3].

This work discusses the rationality of a sum related to Theta-functions without using any long and complicated methods similar to the study of the periodicity of the binary expansion. A mathematical representation

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and a generalization is made for $f(B)=\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ and $B\geq 2$ which defines a special function related to Theta Functions. In this article, we propose a new approximation for the output values of this function after attempting to find the nature of these outputs. Then we prove the continuity of the same function $f(x)=\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}$ when x is a real number with $x>2$.

Since it is very difficult to find a real number B where $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ is a rational number, then this short work is very useful for many fields of applied mathematics and especially for the study of Theta-functions [4] , the summation of Poisson and for the study of the related Riemann Zeta function. This work can also be completed by the article that proposes new expressions for the studied series [5] which makes the findings of this paper more interesting.

2. Representation of the series to understand the rationality

The series $\sum_{n\in\mathbb{N}}(\frac{1}{2})^{n}$ is a sum of terms of unity divided by a power of 2 and everybody knows that $\sum_{n\in\mathbb{N}}(\frac{1}{2})^{n}=2$ which is a rational number. Therefore, an idea that may seem rational for many students is to think that $\sum_{n\in\mathbb{N}}(\frac{1}{2})^{n^2}$ is maybe a rational number or, at least, this sum is only a recurrent decimal since it is also a sum of terms of unity divided by a power of 2. However, the representation of these kind of series by using the sum of Poisson makes it difficult to decide if such series have a rational limit or not.

Let's start by using the following formula of Poisson which is also used for the proof of Riemann Zeta Function in order to make a different representation for the series $\sum_{n\in\mathbb{N}}(\frac{1}{2})^{n^2}$.

The formula of Poisson is: $\sum_{n\in\mathbb{Z}}\exp(-\pi\times n^2\times s)=\frac{1}{\sqrt{s}}\times\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi\times n^2}{s})$ (for any : s >0) (1)

Let's consider that ***t>1*** and let's use ***s=ln(t)*** in this formula. (2)

We have :

$$\sum_{n\in\mathbb{Z}}\exp(-\pi\times n^2\times\ln(t))=\frac{1}{\sqrt{\ln(t)}}\times\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi\times n^2}{\ln(t)}) \quad (3)$$

$$\Leftrightarrow\sum_{n\in\mathbb{Z}}t^{-\pi\times n^2}=\frac{1}{\sqrt{\ln(t)}}\times\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi\times n^2}{\ln(t)}) \quad (4)$$

Now let's consider that : $u=\frac{1}{t}$ and we have consequently : 0<u<1 . (5)

And we have :

$$2\times\sum_{n\in\mathbb{N}}u^{\pi\times n^2}-1=\frac{1}{\sqrt{-\ln(u)}}\times\sum_{n\in\mathbb{Z}}\exp(\frac{\pi\times n^2}{\ln(u)}) \quad (6)$$

Hence : $\sum_{n\in\mathbb{N}}u^{\pi\times n^2}=\frac{1}{2\times\sqrt{-\ln(u)}}\times(\sum_{n\in\mathbb{Z}}\exp(\frac{\pi\times n^2}{\ln(u)}))+\frac{1}{2}$ (7)

Now let's consider that : $u=(\frac{1}{2})^{\frac{1}{\pi}}$ (8)

We can verify that : 0<u<1 .

Consequently, we get this formula :

$$\sum_{n\in\mathbb{N}}(\frac{1}{2})^{n^2}=\frac{1}{2\times\sqrt{\frac{\ln(2)}{\pi}}}\times(\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi^2\times n^2}{\ln(2)}))+\frac{1}{2} \quad (9)$$

And thus, it is easy to remark that only a well developed mathematical demonstration can prove if $\sum_{n\in\mathbb{N}}(\frac{1}{2})^{n^2}$ is rational or not.

Also, we can generalize the result above by considering a real number $B \geq 2$ when applying the formula of Poisson with : $u = (\frac{1}{B})^{\frac{1}{\pi}}$. And we get consequently:

$$\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2} = \frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}} \times (\sum_{n\in\mathbb{Z}} \exp(\frac{-\pi^2\times n^2}{\ln(B)})) + \frac{1}{2} \quad (10)$$

In order to have $\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2}$ rational, the coefficient $\frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}}$ and the series $\sum_{n\in\mathbb{Z}} \exp(\frac{-\pi^2\times n^2}{\ln(B)})$ should be both rational or at least both irrational. However, even that coefficient seems to be rarely rational. Hence, it is even difficult to find a real number ***B*** which respects the fact that: $\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2}$ is rational.

Furthermore, as it is demonstrated in a personal different article [5], this function expressed as series $f(B) = \sum_{n=1}^{n=+\infty} (\frac{1}{B^{n^2}})$ can be expressed also for $B \geq 2$ with :

$$\sum_{n=1}^{n=+\infty} (\frac{1}{B^{n^2}}) = -\sum_{i=0}^{i=+\infty} (\sum_{n=1}^{n=+\infty} (\frac{(-1)^n}{(B^{2^{2i}\times n^2}\times 2^{-i})})) = \frac{1-B}{B} \times \lim_{k\to+\infty} ((\frac{1}{B})^k \times \sum_{i=0}^{i=k} (\sum_{n=1}^{n=+\infty} (\frac{(-1)^n}{(B^{2^{2i}\times n^2 - i}\times 2^{-i})}))) \quad (11)$$

Without forgetting that the notation «***lim*** » is maintained in the formula (11) since the increasing variable **k** should tend to the same value for all the terms even in the «***infinity***».

It is also useful to explore the transcendence demonstrations of the Professor Daniel Bertrand in his article 'Theta functions and transcendence' [4], where he links the Theta-3 Functions to these kind of series as follows:

$$\theta_3(q) = 1+2\times\sum_{n\geq 1} q^{n^2} = (2\times\sum_{n\in\mathbb{N}} (q)^{n^2})-1 = (2\times\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2})-1 \quad \text{with : } B = \frac{1}{q} \quad (12)$$

Unfortunately, the representations of the studied series by the formula (11) doesn't seem to simplify the study of rationality of the series limit. However, we will make an attempt to find new information about the rationality by using the formula of Poisson.

3. An attempt to explore the rationality of the studied series

We have: $$\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2} = \frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}} \times (\sum_{n\in\mathbb{Z}} \exp(\frac{-\pi^2\times n^2}{\ln(B)})) + \frac{1}{2} \quad \text{where } B \geq 2 \quad (13)$$

If $\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2}$ is rational then we have:

$$\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2} = \frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}} \times (\sum_{n\in\mathbb{Z}} \exp(\frac{-\pi^2\times n^2}{\ln(B)})) + \frac{1}{2} = \frac{p}{q} \quad \text{with } p \text{ and } q \in \mathbb{N}\setminus\{0\} \ . \quad (14)$$

And if the coefficient: $\frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}}$ is rational then we have the value: $\sqrt{\frac{\ln(B)}{\pi}}$ is rational.

This means that if the coefficient: $\frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}}$ is rational then:

$$\sqrt{\frac{\ln(B)}{\pi}} = \frac{a}{b} \quad a \text{ and } b \in \mathbb{N}\setminus\{0\} \quad (15)$$

consequently: $$B = \exp(\frac{a^2}{b^2}\times\pi) \quad (16)$$

Now let's consider that: $\sum_{n\in\mathbb{N}} (\frac{1}{B})^{n^2}$ is rational and that: $\frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}}$ is also rational :

We have in this case: $$\sum_{n\in\mathbb{Z}} \exp(\frac{-\pi^2\times n^2}{\ln(B)}) = 2\times\sum_{n\in\mathbb{N}} \exp(\frac{-\pi^2\times n^2}{\ln(B)}) - 1 \quad (17)$$

And we have: $$\sum_{n\in\mathbb{N}} \exp(\frac{-\pi^2\times n^2}{\ln(B)}) = \frac{1}{2}\times\sum_{n\in\mathbb{Z}} \exp(\frac{-\pi^2\times n^2}{\ln(B)}) + \frac{1}{2} \quad (18)$$

And also: $\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=2\sqrt{\frac{\ln(B)}{\pi}}\times(\frac{p}{q}-\frac{1}{2})=\frac{2\times a}{b}\times(\frac{p}{q}-\frac{1}{2})$ (19)

And thus, we have: $\sum_{n\in\mathbb{N}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=\frac{a}{b}\times(\frac{p}{q}-\frac{1}{2})+\frac{1}{2}$ (20)

However, we can notice that:

$$\sum_{n\in\mathbb{N}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=\sum_{n\in\mathbb{N}}\frac{1}{\exp(\frac{\pi^2\times n^2}{\ln(B)})}=\sum_{n\in\mathbb{N}}(\frac{1}{\exp(\frac{\pi^2}{\ln(B)})})^{n^2} \quad (21)$$

Hence, if we consider that : $B'=\exp(\frac{\pi^2}{\ln(B)})$ (22)

Then we have $B'=\exp(\frac{\pi^2}{\ln(B)})>1$ since $\ln(B)>0$. (23)

And thus we can consider that: $\sum_{n\in\mathbb{N}}(\frac{1}{\exp(\frac{\pi^2}{\ln(B)})})^{n^2}=\sum_{n\in\mathbb{N}}(\frac{1}{B'})^{n^2}$ (24)

And we can use the formula of Poisson again. Hence we get:

$$\sum_{n\in\mathbb{N}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=\sum_{n\in\mathbb{N}}(\frac{1}{B'})^{n^2}=\frac{1}{2\times\sqrt{\frac{\ln(\exp(\frac{\pi^2}{\ln(B)}))}{\pi}}}\times(\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi^2\times n^2}{\ln(\exp(\frac{\pi^2}{\ln(B)}))}))+\frac{1}{2} \quad (25)$$

And thus, we get: $\sum_{n\in\mathbb{N}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=\frac{1}{2\times\sqrt{\frac{\pi}{\ln(B)}}}\times(\sum_{n\in\mathbb{Z}}\exp(-n^2\times\ln(B)))+\frac{1}{2}$ (26)

And since we considered that: $B=\exp(\frac{a^2}{b^2}\times\pi)$ then we get:

$$\sum_{n\in\mathbb{N}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=\frac{1}{2\times\frac{b}{a}}\times(\sum_{n\in\mathbb{Z}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi))+\frac{1}{2} \quad (27)$$

We calculated above that: $\sum_{n\in\mathbb{N}}\exp(\frac{-\pi^2\times n^2}{\ln(B)})=\frac{a}{b}\times(\frac{p}{q}-\frac{1}{2})+\frac{1}{2}$ (28)

Hence: $\frac{1}{2\times\frac{b}{a}}\times(\sum_{n\in\mathbb{Z}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi))+\frac{1}{2}=\frac{a}{b}\times(\frac{p}{q}-\frac{1}{2})+\frac{1}{2}$ (29)

Consequently: $\sum_{n\in\mathbb{Z}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi)=\frac{2p}{q}-1$ (30)

And we know that: $\sum_{n\in\mathbb{Z}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi)=2\times\sum_{n\in\mathbb{N}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi)-1$ (31)

And thus we have in this case: $\sum_{n\in\mathbb{N}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi)=\frac{p}{q}$ (32)

Consequently, if we consider that: $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ is rational and that: $\frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}}$ is rational, then we prove that: $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}=\frac{1}{2\times\sqrt{\frac{\ln(B)}{\pi}}}\times(\sum_{n\in\mathbb{Z}}\exp(\frac{-\pi^2\times n^2}{\ln(B)}))+\frac{1}{2}=\sum_{n\in\mathbb{N}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi)$ (33)

with: a and $b\in\mathbb{N}\setminus\{0\}$

However, we should stop here since we remark that we already have:

$$\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}=\sum_{n\in\mathbb{N}}(\frac{1}{\exp(\ln(B))})^{n^2}=\sum_{n\in\mathbb{N}}(\frac{1}{\exp(\frac{a^2}{b^2}\times\pi)})^{n^2}=\sum_{n\in\mathbb{N}}\exp(\frac{-n^2\times a^2}{b^2}\times\pi) \quad (34)$$

And thus, we remark that we return back to the initial formula after applying the formula of Poisson twice. Consequently, if we consider that: $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ is rational and that:

$\exists a$ and $b\in\mathbb{N}\setminus\{0\}$ with $B=\exp(\frac{a^2}{b^2}\times\pi)$ then we can't prove any new formula and our calculations lead us to an infinite loop.

4. A method to approximate the limit of the series

Since it is difficult to find sufficient information about the rationality of the series $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ then we will propose a method to approximate the value of the series.

We have: $B^{i^2} > B^{i\times M} \Leftrightarrow \frac{1}{B^{i^2}} < \frac{1}{B^{i\times M}}$ where $\boldsymbol{M}$ and $\boldsymbol{i}$ are natural numbers and $i > M$. (35)

Hence, we have: $\sum_{i=M}^{n} \frac{1}{B^{i^2}} < \sum_{i=M}^{n} \frac{1}{B^{i\times M}}$ (36)

And thus: $\sum_{i=M}^{+\infty} \frac{1}{B^{i^2}} < \sum_{i=M}^{+\infty} \frac{1}{B^{i\times M}}$ (37)

The order is strict despite the order limit theorem.

We can remark that: $\sum_{i=M}^{+\infty} \frac{1}{B^{i\times M}} = \sum_{i=M}^{+\infty}(\frac{1}{B^M})^i$ (38)

which have the form of geometric series with ratio $\boldsymbol{r}$: $r = \frac{1}{B^M}$ (39)

And that: $\sum_{i=M}^{+\infty}(\frac{1}{B^M})^i = \sum_{i=0}^{+\infty}(\frac{1}{B^M})^i - \sum_{i=0}^{M-1}(\frac{1}{B^M})^i$ (40)

and it is sufficient that ***M>1*** and $\boldsymbol{i}$ variates.

And since: $\sum_{i=0}^{+\infty}(\frac{1}{B^M})^i = \frac{B^M}{B^M-1}$ (41)

And: $\sum_{i=0}^{M-1}(\frac{1}{B^M})^i = \frac{1-\frac{1}{B^{M^2}}}{1-\frac{1}{B^M}} = \frac{B^M - B^{M-M^2}}{B^M-1}$ (42)

Then we have: $\sum_{i=M}^{+\infty} \frac{1}{B^{i^2}} < \frac{B^M}{B^M-1} - \frac{B^M - B^{M-M^2}}{B^M-1}$ (43)

And thus, we get: $\sum_{i=M}^{+\infty} \frac{1}{B^{i^2}} < \frac{B^{M-M^2}}{B^M-1}$ (44)

Finally, we have: $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2} < \frac{B^{M-M^2}}{B^M-1} + \sum_{i=0}^{M-1} \frac{1}{B^{i^2}}$ with ***M>1*** (45)

This result may be very useful for computing the values of the limits of the series $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ which can even help to study the rationality of this series. However, the best method to use this result as an approximation is by using this theorem which we proved:

Theorem 4.1. $\sum_{i=0}^{M-1} \frac{1}{B^{i^2}} < \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2} < \frac{B^{M-M^2}}{B^M-1} + \sum_{i=0}^{M-1} \frac{1}{B^{i^2}}$ *with* ***M>1*** *(46)*

And generally, it is easy to conclude that for any finite natural number $\boldsymbol{K}$, we have this lemma:

Lemma 4.1. $\sum_{i=0}^{M-1} \frac{1}{B^{i^2+K}} < \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2+K} < \frac{B^{M-M^2}}{B^K\times(B^M-1)} + \sum_{i=0}^{M-1} \frac{1}{B^{i^2+K}}$ *with* ***M>1*** *(47)*

And we remark that these approximations proved above become better when we increase the value of the natural number $\boldsymbol{M}$.

5. Discussing the discontinuity of the studied Theta Function

If we consider because of the aspect of the formula of Poisson that the limit of $\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ is not a rational number when $\boldsymbol{B}$ is a real number with $B \geq 2$, then this means that the function $f(B) = \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ when $\boldsymbol{B}$ is a real number with $B \geq 2$ will be a real function that gives only irrational numbers. We would say that the image by the function $\boldsymbol{f}$ of the positive real number line that starts from the origin 2 is only a not continuous set of points. This means therefore that this function $\boldsymbol{f}$ is not continuous at all in any point of its curve when $B \geq 2$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. However, let's prove this result mathematically:

First, we start by proving that the function $f(B) = \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ when $\boldsymbol{B}$ is a real number with $B \geq 2$ is strictly monotone. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be real numbers with $a > b \geq 2$ ***:***

We have: $a > b \Rightarrow \frac{1}{a} < \frac{1}{b} \Rightarrow (1+\frac{1}{a})-(1+\frac{1}{b}) < 0$ (48)

And we have: $a > b \Rightarrow \frac{1}{a^{m^2}} < \frac{1}{b^{m^2}}$ for any natural number ***m*** with ***m>1.*** (49)

Hence we have: $a > b \Rightarrow \sum_{m=2}^{M} \frac{1}{a^{m^2}} < \sum_{m=2}^{M} \frac{1}{b^{m^2}}$ for any natural number ***M with M>m.*** (50)

And thus: $a > b \Rightarrow \sum_{m=2}^{+\infty} \frac{1}{a^{m^2}} - \sum_{m=2}^{+\infty} \frac{1}{b^{m^2}} \leq 0$ (51)

And we have: $1+\frac{1}{a}+\sum_{m=2}^{+\infty} \frac{1}{a^{m^2}} = \sum_{m=0}^{+\infty} \frac{1}{a^{m^2}}$ and: $1+\frac{1}{b}+\sum_{m=2}^{+\infty} \frac{1}{b^{m^2}} = \sum_{m=0}^{+\infty} \frac{1}{b^{m^2}}$ (52)

Consequently: $a > b \Rightarrow \sum_{m=0}^{+\infty} \frac{1}{a^{m^2}} - \sum_{m=0}^{+\infty} \frac{1}{b^{m^2}} < 0 \Rightarrow \sum_{m=0}^{+\infty} \frac{1}{a^{m^2}} < \sum_{m=0}^{+\infty} \frac{1}{b^{m^2}}$ (53)

This proves therefore that the function $f(B) = \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ when ***B*** is a real number with $B \geq 2$ is strictly monotone.

It is sufficient to remark that if the studied Function ***f***is continuous in any closed interval included in $I = [2; +\infty[$ then this function is bijective in any closed interval included in $I = [2; +\infty[$ since we have also $f(B) = \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ strictly monotone. However, if the set of outputs of this function is made only of irrationals, then rational numbers included in any closed interval included in $I = [2; +\infty[$ can never be images by this function as demonstrated above. And thus, this is a contradiction with the fact that the studied Function ***f***is bijective in any closed interval included in $I = [2; +\infty[$. This means therefore that this function is not continuous.

We can also prove the discontinuity in this case of irrationality by using the definition of continuity in term of sequences. Let's consider that we have this sequence $A_n = (a - frac1n)$ where we have $\lim_{n\to+\infty} A_n = (a - frac1n) = a$. (54)

Let's consider that $\lim_{n\to+\infty} f(A_n) = \frac{10}{7}$ and that ***a***is included in a closed interval included in $I = [2; +\infty[$. We chose $\frac{10}{7}$ as an example of rational number as limit of the studied function ***f*** and ***n***is any natural strictly positive number. If you say that the studied Function ***f***is continuous in any closed interval included in $I = [2; +\infty[$ then you should know that this function ***f***is also strictly monotone. The studied Function ***f***is consequently bijective in any closed interval included in $I = [2; +\infty[$ and it should respect the definition of continuity in term of sequences.

Hence, $\lim_{n\to+\infty} f(A_n) = \frac{10}{7}$ should be a consequence of $\lim_{n\to+\infty} A_n = (a - frac1n) = f^{-1}(frac107)$
(55)

since this function ***f***should take all possible values, irrational or not, in any bounded interval in its domain of definition. However, we considered that the set of output values of this studied Function is made only of irrationals. This means that there exists no point ***a***of the domain of definition of this function with $f(a) = frac107$ since $f(a)$ can only be an irrational number from the set of outputs of the studied function of irrationals. And thus, we proved that $\lim_{n\to+\infty} f(A_n) = \frac{10}{7}$ can't be equal to $f(a)$ even if we considered that $\lim_{n\to+\infty} A_n = a$. This means that: $\lim_{n\to+\infty} A_n = a$ and $\lim_{n\to+\infty} f(A_n) \neq f(a)$ and thus the definition of continuity in term of sequences can't be verified and respected for this studied Function ***f*** if the set of its ouputs is made only of irrationals.

And this means also in this case that the related Theta Function $\theta_3(q)$ is not continuous since it is defined by:

$$\theta_3(q) = 1+2\times\sum_{n\geq 1} q^{n^2} = (2\times\sum_{n\in\mathbb{N}}(q)^{n^2})-1 = (2\times\sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2})-1 \text{ with : } B=\frac{1}{q} \quad (56)$$

6. The proof of continuity of the studied Theta Function

We can obviously remark that the curve of the function**f** , or of a part of it, seems continuous by observing the curve of this function on MATLAB or any other Analysis and calculus software. Let's make therefore an investigation of continuity without using the formula of Poisson.

Let's start by proving the continuity of the function $f(x) = \sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}$ at the right of the real point $\boldsymbol{a}$ with $a > 2$. Let's use the mathematical definition of the continuity of real functions.

The definition of the continuity in this case is:

$$\forall\varepsilon > 0\ \exists\eta > 0\ \forall x \in I\ (|x-a| \leq \eta \Rightarrow |f(x)-f(a)| \leq \varepsilon) \quad (57)$$

Hence, since x>a then the definition becomes:

$$\forall\varepsilon > 0\ \exists\eta > 0\ \forall x \in I\ (x-a \leq \eta \Rightarrow f(a)-f(x) \leq \varepsilon) \quad (58)$$

And thus, we get the following definition:

$$\forall\varepsilon > 0\ \exists\eta > 0\ \forall x \in I\ (x-a \leq \eta \Rightarrow \sum_{N\in\mathbb{N}}(\tfrac{1}{a})^{N^2} - \sum_{N\in\mathbb{N}}(\tfrac{1}{x})^{N^2} \leq \varepsilon) \quad (59)$$

Let's consider that: $\eta' = \frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(x\times a)^{n^2}}{\sum_{i=0}^{n^2-1} x^{n^2-1-i}\times a^i}$ (60)

where n can be any natural number with $n \geq 2$

Let's consider that: $\eta = \frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(2\times a)^{n^2}}{\sum_{i=0}^{n^2-1} 2^{n^2-1-i}\times a^i}$ (61)

where n can be any natural number with $n \geq 2$.

we have: $x > 2 \Rightarrow \eta < \eta'$ (62)

Now let's verify:

We have: $x-a \leq \eta \Rightarrow x-a \leq \eta' \Rightarrow x-a \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(x\times a)^{n^2}}{\sum_{i=0}^{n^2-1} x^{n^2-1-i}\times a^i}$ (63)

Hence: $\frac{(x-a)\times\sum_{i=0}^{n^2-1}(x^{n^2-1-i}\times a^i)}{(x\times a)^{n^2}} \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)$ (64)

And thus, we have: $\frac{x^{n^2}-a^{n^2}}{(x\times a)^{n^2}} \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)$ (65)

Consequently, we get: $\frac{1}{a^{n^2}} - \frac{1}{x^{n^2}} \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)$ (66)

However, we should remark also that we have: $\sum_{n=2}^{m}(\frac{1}{a^{n^2}} - \frac{1}{x^{n^2}}) \leq \sum_{n=2}^{m}(\frac{1}{2^n}\times 2\times(\varepsilon-1))$ (67)

Hence: $\sum_{n=2}^{m}\frac{1}{a^{n^2}} - \sum_{n=2}^{m}\frac{1}{x^{n^2}} \leq 2\times(\varepsilon-1)\times\sum_{n=2}^{m}\frac{1}{2^n}$ (68)

Let's remark that: $\sum_{n=0}^{m}\frac{1}{a^{n^2}} - \sum_{n=0}^{m}\frac{1}{x^{n^2}} = \frac{1}{a} - \frac{1}{x} + \sum_{n=2}^{m}\frac{1}{a^{n^2}} - \sum_{n=2}^{m}\frac{1}{x^{n^2}}$ (69)

And we know that: $\frac{1}{a}-\frac{1}{x}=\frac{x-a}{a\times x}\leq 1$ (70)

Consequently: $\sum_{n=0}^{m}\frac{1}{a^{n^2}}-\sum_{n=0}^{m}\frac{1}{x^{n^2}}\leq 1+2\times(\varepsilon-1)\times\sum_{n=2}^{m}\frac{1}{2^n}$ (71)

And thus: $\sum_{n=0}^{+\infty}\frac{1}{a^{n^2}}-\sum_{n=0}^{+\infty}\frac{1}{x^{n^2}}\leq 1+2\times(\varepsilon-1)\times\sum_{n=2}^{+\infty}\frac{1}{2^n}$ (72)

And since: $\sum_{n=2}^{+\infty}\frac{1}{2^n}=\frac{1}{2}$ (73)

We conclude that: $\sum_{n=0}^{+\infty}\frac{1}{a^{n^2}}-\sum_{n=0}^{+\infty}\frac{1}{x^{n^2}}\leq\varepsilon$ (74)

This means that: $x-a\leq\eta\Rightarrow\sum_{n=0}^{+\infty}\frac{1}{a^{n^2}}-\sum_{n=0}^{+\infty}\frac{1}{x^{n^2}}\leq\varepsilon$ (75)

Finally, we conclude that we have indeed:

$\forall\varepsilon>0\ \exists\eta>0\ \forall x\in I\ (x-a\leq\eta\Rightarrow\sum_{N\in\mathbb{N}}(\frac{1}{a})^{N^2}-\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}\leq\varepsilon)$ (76)

With: $\eta=\frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(2\times a)^{n^2}}{\sum_{i=0}^{n^2-1}2^{n^2-1-i}\times a^i}$ where n can be any natural number with $n\geq 2$.
(77)

This means that the function $f(x)=\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}$ when x is a real number with $x>2$ is continuous at the right in the domain $I=]2;+\infty[$.

We can follow the same steps in order to prove the continuity of the function of irrationals f at the left of the point ***a***which is in the domain $I=]2;+\infty[$.

Since x<a then the definition becomes:

$\forall\varepsilon>0\ \exists\eta>0\ \forall x\in I\ (a-x\leq\eta\Rightarrow f(x)-f(a)\leq\varepsilon)$ (78)

And thus, we get the following definition:

$\forall\varepsilon>0\ \exists\eta>0\ \forall x\in I\ (a-x\leq\eta\Rightarrow\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}-\sum_{N\in\mathbb{N}}(\frac{1}{a})^{N^2}\leq\varepsilon)$ (79)

We consider here that: $\eta'=\frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(x\times a)^{n^2}}{\sum_{i=0}^{n^2-1}a^{n^2-1-i}\times x^i}$ (80)

where n can be any natural number with $n\geq 2$

We consider also that: $\eta=\frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(2\times a)^{n^2}}{\sum_{i=0}^{n^2-1}a^{n^2-1-i}\times 2^i}$ (81)

where n can be any natural number with $n\geq 2$

we have: $x>2\Rightarrow\eta<\eta'$ (82)

Now let's verify:

We have: $a-x\leq\eta\Rightarrow a-x\leq\eta'\Rightarrow a-x\leq\frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(x\times a)^{n^2}}{\sum_{i=0}^{n^2-1}a^{n^2-1-i}\times x^i}$ (83)

Hence: $\frac{(a-x)\times\sum_{i=0}^{n^2-1}(a^{n^2-1-i}\times x^i)}{(x\times a)^{n^2}} \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)$ (84)

And thus, we have: $\frac{a^{n^2}-x^{n^2}}{(x\times a)^{n^2}} \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)$ (85)

Consequently, we get: $\frac{1}{x^{n^2}} - \frac{1}{a^{n^2}} \leq \frac{1}{2^n}\times 2\times(\varepsilon-1)$ (86)

And by following the same steps, we get: $\sum_{n=0}^{+\infty}\frac{1}{x^{n^2}} - \sum_{n=0}^{+\infty}\frac{1}{a^{n^2}} \leq 1+2\times(\varepsilon-1)\times\sum_{n=2}^{+\infty}\frac{1}{2^n}$ (87)

We conclude that: $\sum_{n=0}^{+\infty}\frac{1}{x^{n^2}}-\sum_{n=0}^{+\infty}\frac{1}{a^{n^2}} \leq \varepsilon$ since: $\sum_{n=2}^{+\infty}\frac{1}{2^n} = \frac{1}{2}$ (88)

This means that: $a-x \leq \eta \Rightarrow \sum_{n=0}^{+\infty}\frac{1}{x^{n^2}}-\sum_{n=0}^{+\infty}\frac{1}{a^{n^2}} \leq \varepsilon$ (89)

Finally, we conclude that we have indeed:

$\forall\varepsilon>0\ \exists\eta>0\ \forall x\in I\ (a-x\leq\eta\Rightarrow\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}-\sum_{N\in\mathbb{N}}(\frac{1}{a})^{N^2}\leq\varepsilon)$ (90)

With: $\eta = \frac{1}{2^n}\times 2\times(\varepsilon-1)\times\frac{(2\times a)^{n^4}}{\sum_{i=0}^{n^4-1}a^{n^4-1-i}\times 2^i}$ where n can be any natural number with $n\geq 2$. (91)

This means that the function $f(x)=\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}$ when x is a real number with $x>2$ is continuous at the left in the domain $I=]2;+\infty[$.

We conclude that the function $f(x)=\sum_{N\in\mathbb{N}}(\frac{1}{x})^{N^2}$ when x is a real number with $x>2$ is continuous in all the domain $I=]2;+\infty[$.

Finally, we conclude that the related Theta Function $\theta_3(q)$ is continuous in all the domain $I=]2;+\infty[$.

7. Summary and conclusion

The article started by describing how difficult is to guess or even prove if the values of the studied function $f(B) = \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ is rational or irrational. Hence, each real value $\boldsymbol{B}$ can be studied alone by using mathematical tools similar to the approximation demonstrated in this work which is:

$$\sum_{i=0}^{M-1}\frac{1}{B^{i^2+K}} < \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2+K} < \frac{B^{M-M^2}}{B^K\times(B^M-1)} + \sum_{i=0}^{M-1}\frac{1}{B^{i^2+K}}$$

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are finite natural numbers with $\boldsymbol{M>1}$ and where the approximation becomes better when we increase the value of $\boldsymbol{M}$.

This work also proves the continuity of this function $\boldsymbol{f}$ with the definition after demonstrating that if the set of outputs of this function is purely irrational, or purely rational, then this contradicts the continuity of this studied function $\boldsymbol{f}$.

Finally, since it is very difficult to find a real number $\boldsymbol{y}$ where $\sum_{n\in\mathbb{N}}(\frac{1}{y})^{n^2}$ is a rational number, then I still invite the readers to discuss the rationality of the function $f(B) = \sum_{n\in\mathbb{N}}(\frac{1}{B})^{n^2}$ because proving mathematically its irrationality in any interval will threaten the use of the summation of Poisson which is widely applied in many fields especially for the proof of Riemann Zeta Function. Unfortunately, the article 'Theta functions and transcendence' [4] of Professor Bertrand gives only results about the transcendence of algebraic numbers whereas we need results about whole intervals of real numbers.

Finally, I really appreciated that my revised article has been peer reviewed and accepted by the journal "Asia Mathematika'" which accepts similar researches related to the continuity and topology [6, 7, 8].

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