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An applicable generalization of fractional quantum differential equations with arbitrary fractional order via multi-step methods

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Abstract: This paper studies a nonlinear multi term fractional q-differential equations of arbitrary fractional orders. New existence and uniqueness comes about are built up utilizing Banach contraction principle. Other existence results are gotten utilizing fixed point theorems of Schaefer and Krasnoselskii. In order to clarify our results, some illustrative examples are also shown.

Key words: Arbitrary fractional order; Caputo q–derivation; multi-step methods; nonlinear analysis; numerical direction

1. Introduction

Jackson in [1] introduced the quantum calculus. Then it developed by Al-Salam who started fitting the concept of q–fractional calculus [2]. Agarwal continued studying certain q–fractional integrals and derivatives [3]. Further, some researchers studied q–difference equations; see [4–14] for more details. On other side, fractional differential equations ($\mathbb{FDE}$) have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc.

El-Seyed discussed a class of nonelinear $\mathbb{FDE}$ of arbitrary orders [15]. Lakshmikantham initiated the basic theory for fractional functional differential equations [16]. In 2005, Bai *et al.* presented the boundary problem $\mathbb{D}_0^{\beta}\mathfrak{m}(v) = \mathrm{w}(v, \mathfrak{m}(v))$, under conditions $\mathfrak{m}(0) = \mathfrak{m}(1) = 0$, where $v \in \Delta := (0, 1)$ and $0 < \beta \leq 2$ [17]. In 2008, Qiu *et al.* studied the equation with conditions $\mathfrak{m}(0) = \mathfrak{m}'(1) = \mathfrak{m}''(1) = 0$, where $v \in \Delta$, $2 < \beta < 3$ and $\mathrm{w} : \overline{\Delta} \times [0, \infty) \to [0, \infty)$ is such that $\lim_{v\to 0^+} \mathrm{w}(v, .) = \infty$, here $\overline{\Delta} := [0, 1]$ [18]. In 2010, Agarwal *et al.* considered the singular fractional Dirichlet problem:

$$\mathbb{D}^{\beta}\mathfrak{m}(v) + \mathrm{w}(v, \mathfrak{m}(v), \mathbb{D}^{\gamma}\mathfrak{m}(v)) = 0, \quad \beta \in (1, 2], \gamma > 0, \beta - \gamma \geq 1, \tag{1}$$

with boundary value problem $\mathfrak{m}(0) = \mathfrak{m}(1) = 0$, where $\mathrm{w} \in \mathrm{Car}(\overline{\Delta} \times \mathcal{B})$ with $\mathcal{B} = (0, \infty) \times \mathbb{R}$, w is positive and singular at $v = 0$ [19]. The non-constant real-valued function $\mathfrak{m}$ on the interval $I = [\iota_1, \iota_2]$ is said to be singular on I, if it is continuous and exists a set $S \subseteq I$ with measure 0, the derivative of y exists and and be

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zero on complement of S. That is, the derivative of $\mathfrak{m}$ vanish almost everywhere and $\mathfrak{m}$ is non-constant on I. In 2012, Cabada *et al.* investigated the positive solutions' existence for the nonlinear $\mathbb{FDE}$:

$$\begin{cases} \mathbb{D}^{\beta}\mathfrak{m}(t) = \mathrm{w}(v, \mathfrak{m}(v)) \\ \mathfrak{m}(0) = \mathfrak{m}''(1) = 0,\ \mathfrak{m}(1) = \int_0^1 \mathfrak{m}(\xi)\,\mathrm{d}\xi, \end{cases} \tag{2}$$

here $v \in \Delta$, $2 < \beta < 3$ and continuous function $\mathrm{w} : \overline{\Delta} \times [0, \infty) \to [0, \infty)$ [20]. In 2014, Li reviewed the Problem (1), under conditions $\mathfrak{m}(0) = \mathfrak{m}'(1) = 0$ and $\mathfrak{m}'(1) = \mathbb{D}^{\beta}\mathfrak{m}(v)$, where $0 < v < 1$, $\beta \in (2, 3)$, $0 < \gamma < 1$, $\mathrm{w} : (0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function that may have singularity at $v = 0$ [21]. In 2016, the fractional integro-differential equation:

$$\mathbb{D}^{\gamma}\mathfrak{m}(v) = \mathrm{w}\left(v, \mathfrak{m}(v), \mathfrak{m}'(v), \mathbb{D}^{\alpha}\mathfrak{m}(v), \mathbb{I}^{\beta}\mathfrak{m}(v)\right), \tag{3}$$

under conditions $\mathfrak{m}'(0) = \mathfrak{m}(\eta)$, $\mathfrak{m}(1) = \int_0^{\nu} \mathfrak{m}(\eta)\,\mathrm{d}\eta$ and $\mathfrak{m}^{(i)}(0) = 0$ for $i = 2, \dots, [\gamma] - 1$ was investigated, where $v \in \Delta$, $\gamma \in [2, 3)$, $\mathfrak{m} \in \bar{\mathcal{B}}$, $\alpha, \eta, \nu \in \Delta$, $\beta > 1$ and $\mathrm{w} : \overline{\Delta} \times \mathbb{R}^4 \to \mathbb{R}$ is a function s.t $\mathrm{w}(v, ., ., ., .)$ is singular at some point $v \in \overline{\Delta}$ [22]. Houas *et al.* in [23] considered the following $\mathbb{FDE}$:

$$\mathbb{D}^{\zeta_0}\mathfrak{m}(v) = \mathrm{w}\big(v, \mathfrak{m}(v), \mathbb{D}^{\zeta_1}\mathfrak{m}(v), \dots, \mathbb{D}^{\zeta_{n-1}}\mathfrak{m}(v)\big), \qquad v \in \overline{\Delta}, \tag{4}$$

via conditions $\mathfrak{m}(0) = \mathfrak{m}_{\circ}$, $\mathfrak{m}'(0) = \mathfrak{m}''(0) = \cdots = \mathfrak{m}^{(n-2)}(0) = 0$, $\mathbb{I}^{\gamma}\mathfrak{m}(1) = \lambda\,\mathbb{I}^{\gamma}\mathfrak{m}(\xi)$, where $\mathbb{D}^{\zeta_i}$, $i = 0, 1, 2, \dots, n-1$, denote the Caputo fractional derivatives, $\lambda \neq 0$, $\gamma, \xi \in \Delta$ are real real numbers and w is a function with necessary conditions. In 2020, Samei considered the singular system of q-differential equations:

$$\begin{cases} \mathbb{D}_q^{\alpha_1}\mathfrak{m}(v) = \mathrm{w}_1(v, \mathfrak{m}(v), \acute{\mathfrak{m}}(v)), \\ \mathbb{D}_q^{\alpha_2}\acute{\mathfrak{m}}(v) = \mathrm{w}_2(v, \mathfrak{m}(v), \acute{\mathfrak{m}}(v)), \end{cases} \tag{5}$$

with cconditions $\mathfrak{m}(0) = \acute{\mathfrak{m}}(0) = 0$, $\mathfrak{m}^{(i)}(0) = \acute{\mathfrak{m}}^{(i)}(0) = 0$, for $i = 2, \dots, n-1$ and $\mathfrak{m}(1) = \left[\mathbb{I}_q^{\gamma_1}\left(\hbar_1(v)\mathfrak{m}(v)\right)\right]_{v=1}$, $\acute{\mathfrak{m}}(1) = \left[\mathbb{I}_q^{\gamma_2}\left(\hbar_2(v)\acute{\mathfrak{m}}(v)\right)\right]_{v=1}$, Similarly, some related results have been obtained in [24–28]. The authors studied a nonlinear system involving the fractional p, q-Laplacian in $\mathbb{R}^n$

$$\begin{cases} (-\Delta)_p^{s_1}\mathfrak{m}(v) + (-\Delta)_q^{s_2}\mathfrak{m}(v) = \mathrm{w}_1(\mathfrak{m}(v), \mathfrak{n}(v)), \\ (-\Delta)_p^{s_1}\mathfrak{n}(v) + (-\Delta)_q^{s_2}\mathfrak{n}(v) = \mathrm{w}_2(\mathfrak{m}(v), \mathfrak{n}(v)), \end{cases} \tag{6}$$

for $v \in \mathbb{R}^n$ where $\mathfrak{m}, \mathfrak{n} > 0$, $0 < s_1, s_2 < 1$, $p, q > 2$ [29]. Liu *et al.* we investigated the anisotropic (p, q)-Robinp roblem

$$(-\Delta)_{p(v)}\mathfrak{m}(v) + (-\Delta)_q\mathfrak{m}(v) + \alpha(v)\left(\mathfrak{m}(v)\right)^{p(v)-1} = \lambda\left(\mathfrak{m}(v)\right)^{\tau(v)-1} + \mathrm{w}(v, \mathfrak{m}(v)), \tag{7}$$

in $\Omega \subseteq \mathbb{R}^N$ under condition $\frac{\partial \mathfrak{m}}{\partial n_{pq}} + \beta(v)\left(\mathfrak{m}(v)\right)^{p(v)-1} = 0$ on $\partial\Omega$ where $\mathfrak{m}(v) > 0$, $\lambda > 0$ [30, 31]. In 2023, Thabet *et al.* investigated the existence, uniqueness and Hyers-Ulam stability of solutions for fractional nonlinear couple snap system in the $\mathbb{G}$-Caputo sense

$$\begin{cases} {}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathfrak{t}) = \mathrm{u}_1(\mathfrak{t}), \quad {}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathfrak{t}) = \mathrm{u}_2(\mathfrak{t}), \\ {}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\mathrm{u}_1(\mathfrak{t}) = \mathrm{w}_1(\mathfrak{t}), \quad {}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\mathrm{u}_2(\mathfrak{t}) = \mathrm{w}_2(\mathfrak{t}), \\ {}^c\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\mathrm{w}_1(\mathfrak{t}) = \mathrm{x}_1(\mathfrak{t}), \quad {}^c\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\mathrm{w}_2(\mathfrak{t}) = \mathrm{x}_2(\mathfrak{t}), \\ {}^c\mathcal{D}_{\iota_1^+}^{s_1;\mathbb{G}}\mathrm{x}_1(\mathfrak{t}) = h_1(\mathfrak{t}, \mathrm{v}_1, \mathrm{v}_2, \mathrm{u}_1, \mathrm{u}_2, \mathrm{w}_1, \mathrm{w}_2, \mathrm{x}_1, \mathrm{x}_2), \\ {}^c\mathcal{D}_{\iota_1^+}^{s_2;\mathbb{G}}\mathrm{x}_2(\mathfrak{t}) = h_2(\mathfrak{t}, \mathrm{v}_1, \mathrm{v}_2, \mathrm{u}_1, \mathrm{u}_2, \mathrm{w}_1, \mathrm{w}_2, \mathrm{x}_1, \mathrm{x}_2), \end{cases} \tag{8}$$

subject to some integral boundary conditions, where the function $\mathbb{G} \in C^1(\Sigma)$ is increasing with $\mathbb{G}'(\mathfrak{t}) \neq 0$, for all $\mathfrak{t} \in \Sigma = [\iota_1, \iota_2]$ and the functions $h_k \in C(\Sigma \times \mathbb{R}^8), (k = 1, 2)$ and $\mathrm{g}_{kj} \in C(\Sigma, \mathbb{R}), (j = 0, 1, 2, 3;\ k = 1, 2)$ are continuous functions [32]. Also, in generalizing the published article [32, 33], they studied the existence and uniqueness along with various types of Ulam-Hyers stability of solutions for the $\varrho-$Hilfer fractional snap dynamic system on unbounded domains $[a, \infty)$,

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) = \Psi_2(u), \quad {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Psi_2(u) = \Psi_3(u), \quad {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Psi_3(u) = \Psi_4(u), \\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_4,\beta_4;\varrho}\Psi_4(u) = \mathcal{G}\left(u, \Psi_1(u), \Psi_2(u), \Psi_3(u), \Psi_4(u)\right), \end{cases} \tag{9}$$

for $u \in J := [a, \infty)$, $a \geq 0$, under some $\varrho-$Riemann-Liouville fractional integrals conditions where $\lambda_i \in \mathbb{R}, i = 1, \ldots, 4$, $\alpha \in \{\alpha_i\}$ and $\beta \in \{\beta_i\}$, respectively, $\alpha_i \leq \gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, and $\mathcal{G} : J \times \Sigma^4 \to \Sigma$, $\omega_i : J \to \Sigma$, are continuous functions in the real Banach space Σ [34]. For more instance, consider [35–39].

In this work, we consider the quantum type of Problem (4) and deal with existence and uniqueness of solutions for the following multi term nonlinear fractional q-differential equation ($\mathbb{F}q$-$\mathbb{DE}$)

$$\mathbb{D}_q^{\zeta_0}\mathfrak{m}(v) = \mathrm{w}\big(v, \mathfrak{m}(v), \mathbb{D}_q^{\zeta_1}\mathfrak{m}(v), \ldots, \mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m}(v)\big), \qquad v \in \overline{\Delta}, \tag{10}$$

subject to the boundary condition

$$\begin{cases} \mathfrak{m}(0) = \mathfrak{m}_\circ, \\ \mathfrak{m}'(0) = \mathfrak{m}''(0) = \cdots = \mathfrak{m}^{(n-2)}(0) = 0, \\ \mathbb{I}_q^{\gamma}\mathfrak{m}(1) = \lambda\mathbb{I}_q^{\gamma}\mathfrak{m}(\xi), \end{cases} \tag{11}$$

where $\mathbb{D}_q^{\zeta_i}$, $i = 0, 1, 2, \ldots, n-1$, denote the Caputo fractional $q-$derivatives with

$$0 < \zeta_{n-1} < \zeta_{n-2} < \cdots < \zeta_2 < \zeta_1 < \zeta_0 < n,$$

$n \in \{0\} \cup \mathbb{N}$, $\lambda \neq 0$ is real constant, $\mathfrak{m}_\circ \in \mathbb{R}$, $\gamma, \xi \in \Delta$ are real real numbers and w is a function with necessary conditions.

In Sec. 2, we recall some essential definition of quantom fractional derivative and integral. Sec. 3 contains our main results in this work, while an example is presented to support the validity of our obtained results. An application with some needed algorithms for the problems are given in Sec. 4. In Sec. 5, conclusion are presented.

2. Basic concepts and preliminaries

Throughout the context, we shall apply the notations of time scales calculus [40]. The q-fractional calculus is considered here on $\mathbb{T}_{\mathrm{s}_0} = \{0\} \cup \{\mathrm{s} : \mathrm{s} = \mathrm{s}_0 q^{\aleph}\}$, for $\aleph \in \mathbb{N}$, $\mathrm{s}_0 \in \mathbb{R}$ and $q \in \Delta$. If there is no confusion concerning s_0 we shall denote $\mathbb{T}_{\mathrm{s}_0}$ by $\mathbb{T}$.

2.1. Quantum calculus

Let $p \in \mathbb{R}$. Define $[p]_q = (1 - q^p)(1 - q)^{-1}$ [1]. The $q-$factorial function $(\mathrm{v} - \mathrm{w})_q^{(\aleph)}$ with $\aleph \in \mathbb{N}_0$ is defined by

$$(\mathrm{v} - \mathrm{w})_q^{(\aleph)} = \prod_{\mathrm{k}=0}^{\aleph-1}(\mathrm{v} - \mathrm{w}q^{\mathrm{k}}), \qquad \mathrm{v}, \mathrm{w} \in \mathbb{R}, \tag{12}$$

and $(\mathrm{v}-\mathrm{w})_q^{(0)}=1$, where $\mathbb{N}_0:=\{0,1,2,3,\dots\}$ ([4]). Also, for $\sigma\in\mathbb{R}$ and $a\neq 0$, we have

$$(\mathrm{v}-\mathrm{w})_q^{(\sigma)}=\mathrm{v}^\sigma\prod_{k=0}^{\infty}\frac{\mathrm{v}-\mathrm{w}q^k}{\mathrm{v}-\mathrm{w}q^{\sigma+k}}. \tag{13}$$

The $q-$Gamma function is given by $\Gamma_q(z)=(1-q)^{1-z}(1-q)_q^{(z-1)}$, where $z\in\mathbb{R}\backslash\{\cdots,-2,-1,0\}$ [1]. In fact, by using (13), we have $\Gamma_q(z)=(1-q)^{1-z}\prod_{k=0}^{\infty}\frac{1-q^{k+1}}{1-q^{z+k-1}}$. Note that, $\Gamma_q(z+1)=[z]_q\Gamma_q(z)$ [41, Lemma 1]. For a function $w:\mathbb{T}\to\mathbb{R}$, the $q-$derivative of w, is

$$\mathbb{D}_q[\mathfrak{m}](\mathrm{v})=\left(\frac{\mathrm{d}}{\mathrm{d}v}\right)_q\mathfrak{m}(v)=\frac{\mathfrak{m}(v)-\mathfrak{m}(qv)}{(1-q)v}, \tag{14}$$

for all $v\in\mathbb{T}\setminus\{0\}$, and $\mathbb{D}_q[\mathfrak{m}](0)=\lim_{v\to 0}\mathbb{D}_q[\mathfrak{m}](v)$ [4]. Also, the higher order $q-$derivative of the function $\mathfrak{m}$ is defined by $\mathbb{D}_q^n[\mathfrak{m}](v)=\mathbb{D}_q\left[\mathbb{D}_q^{n-1}[\mathfrak{m}]\right](v)$, for all $n\geq 1$, where $\mathbb{D}_q^0[\mathfrak{m}](v)=\mathfrak{m}(v)$ [4]. In fact

$$\mathbb{D}_q^n[\mathfrak{m}](v)=\frac{1}{v^n(1-q)^n}\sum_{k=0}^{n}\frac{\left(1-q^{-n}\right)_q^{(k)}}{(1-q)_q^{(k)}}q^k\mathfrak{m}(vq^k), \tag{15}$$

for $v\in\mathbb{T}\setminus\{0\}$ [5]. The $q-$integral of the function $\mathfrak{m}$ is defined by

$$\mathbb{I}_q[\mathfrak{m}](v)=\int_0^v\mathfrak{m}(\eta)\,\mathrm{d}_q\eta=v(1-q)\sum_{k=0}^{\infty}q^k\mathfrak{m}(vq^k), \tag{16}$$

for $0\leq v\leq b$, provided the series is absolutely converges [4]. If s_1 in $[0,s]$, then

$$\int_{s_1}^{s}\mathfrak{m}(\eta)\,\mathrm{d}_q\eta=\mathbb{I}_q[\mathfrak{m}](s)-\mathbb{I}_q[\mathfrak{m}](s_1)=(1-q)\sum_{k=0}^{\infty}q^k\left[s\mathfrak{m}(sq^k)-s_1\mathfrak{m}(s_1q^k)\right], \tag{17}$$

whenever the series exists. The operator $\mathbb{I}_q^n$ is given by $\mathbb{I}_q^0[\mathfrak{m}](v)=\mathfrak{m}(v)$ and $\mathbb{I}_q^n[\mathfrak{m}](v)=\mathbb{I}_q\left[\mathbb{I}_q^{n-1}[\mathfrak{m}]\right](v)$, for $n\geq 1$ and $g\in C([0,s])$ [4]. It has been proved that $\mathbb{D}_q\left[\mathbb{I}_q[\mathfrak{m}]\right](v)=\mathfrak{m}(v)$ and $\mathbb{I}_q\left[\mathbb{D}_q[\mathfrak{m}]\right](v)=\mathfrak{m}(v)-\mathfrak{m}(0)$, whenever the function $\mathfrak{m}$ is continuous at $v=0$ [4]. The fractional Riemann-Liouville type q-integral of the function $\mathfrak{m}$ is defined by

$$\mathbb{I}_q^\sigma[\mathfrak{m}](v)=\frac{1}{\Gamma_q(\sigma)}\int_0^v(v-\eta)_q^{(\sigma-1)}\mathfrak{m}(\eta)\,\mathrm{d}_q\eta,\qquad \mathbb{I}_q^0[\mathfrak{m}](v)=\mathfrak{m}(v), \tag{18}$$

$\forall\, v\in\overline{\Delta}$ and $\sigma>0$ [5, 42]. The Caputo fractional q-derivative of the function $\mathfrak{m}$ is defined by

$${}^{C}\mathbb{D}_q^\sigma[\mathfrak{m}](v)=\mathbb{I}_q^{[\sigma]-\sigma}\left[\mathbb{D}_q^{[\sigma]}[\mathfrak{m}]\right](v)=\frac{1}{\Gamma_q([\sigma]-\sigma)}\int_0^v(v-\eta)_q^{([\sigma]-\sigma-1)}\,\mathbb{D}_q^{[\sigma]}[\mathfrak{m}](\eta)\,\mathrm{d}_q\eta, \tag{19}$$

$\forall\, v\in\overline{\delta}$ and $\sigma>0$ [42, 43]. It has been proved that $\mathbb{I}_q^\nu\left[\mathbb{I}_q^\sigma[\mathfrak{m}]\right](v)=\mathbb{I}_q^{\sigma+\nu}[\mathfrak{m}](v)$, and ${}^{C}\mathbb{D}_q^\sigma\left[\mathbb{I}_q^\sigma[\mathfrak{m}]\right](v)=\mathfrak{m}(v)$, where $\sigma,\nu\geq 0$ [42]. The authors in [44] presented all Algorithms and MATLAB lines to simplify q-factorial functions $(v-w)_q^{(n)}$, $(v-w)_q^{(\sigma)}$, $\Gamma_q(v)$, $\mathbb{I}_q[\mathfrak{m}](v)$ and some necessary Equations. The following lemmas can be found in [13, 44, 45].

Lemma 2.1. *For $\sigma > 0$, the general solution of the fractional q-differential equation ${}^{C}\mathbb{D}^{\sigma}\mathfrak{m}(v) = 0$ is given by $\mathfrak{m}(v) = \sum_{i=0}^{n-1} e_i v^i$, where $e_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots, n-1$ and $n = [\sigma] + 1$ here $[\sigma]$ denotes the integer part of the real number σ.*

We consider the Banach space $\mathfrak{E} = \left\{\mathfrak{m} \in C(\overline{\Delta}, \mathbb{R}) : \mathbb{D}_q^{\zeta_1}, \mathbb{D}_q^{\zeta_2}, \ldots, \mathbb{D}_q^{\zeta_{n-1}} \in C\left(\overline{\Delta}, \mathbb{R}\right)\right\}$, endowed with the norm $\|\mathfrak{m}\|_{\mathfrak{E}} = \|\mathfrak{m}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m}\| + \|\mathbb{D}_q^{\zeta_2}\mathfrak{m}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m}\|$, where $\|\mathfrak{m}\| = \sup_{v\in\overline{\Delta}} |\mathfrak{m}|$ and $\|\mathbb{D}_q^{\zeta_\iota}\mathfrak{m}\| = \sup_{v\in\overline{\Delta}} |\mathbb{D}_q^{\zeta_\iota}\mathfrak{m}|$ for $\iota = 1, 2, \ldots, n-1$.

Lemma 2.2 ([5])**.** *Let $\zeta, \gamma > 0$, $\mathfrak{m} \in L_1([\iota_1, \iota_2])$. Then $\mathbb{I}_q^{\zeta}\mathbb{I}_q^{\gamma}\mathfrak{m}(v) = \mathbb{I}_q^{\zeta+\gamma}\mathfrak{m}(v)$, $\mathbb{D}_q^{\zeta}\mathbb{I}_q^{\zeta}\mathfrak{m}(v) = \mathfrak{m}(v)$ for each $v \in [\iota_1, \iota_2]$.*

Lemma 2.3 ([5])**.** *Let $\zeta > \gamma > 0$, $\mathfrak{m} \in L_1([\epsilon_1, \epsilon_2])$. Then $\forall v \in [\iota_1, \iota_2]$, $\mathbb{D}_q^{\zeta}\mathbb{I}_q^{\gamma}\mathfrak{m}(v) = \mathbb{I}_q^{\gamma-\zeta}\mathfrak{m}(v)$.*

Lemma 2.4 ([43])**.** *For $\zeta > 0$, the general solution of the $\mathbb{F}q$-$\mathbb{DE}$ $\mathbb{D}_q^{\zeta}\mathfrak{m}(v) = 0$ is $\mathfrak{m}(v) = \sum_{i=1}^{n-1} c_i v^i$, where $n = [\zeta] + 1$ and $c_i \in \mathbb{R}$.*

Lemma 2.5 ([5])**.** *If $\mathfrak{m} \in C(\overline{\Delta}, \mathbb{R}) \cap L_1(\overline{\Delta})$ with $\mathbb{D}_q^{\zeta}\mathfrak{m} \in C(\Delta, \mathbb{R}) \cap L_1(\Delta)$, then $\mathbb{I}_q^{\zeta}\mathbb{D}_q^{\zeta}\mathfrak{m}(v) = \mathfrak{m}(v) + \sum_{i=1}^{n} c_i v^{\zeta-i}$, where $[\zeta] \le n < [\zeta] + 1$ and c_i is some real number.*

2.2. Multi-step methods

As in the case of ordinary differential equations, linear multi-step methods for $\mathbb{FDE}$s makes use of approximations of values of $\mathfrak{m}_i(v)$ $(i = 1, \ldots, n)$ and $\mathrm{W}(v, \mathfrak{m}_1(v), \mathfrak{m}_2(v), \ldots, \mathfrak{m}_n(v))$, on some points of a partition $\eta_0 < \eta_1 < \cdots < \eta_n$ [46, 47]. We will hence compose linear multi-step methods for the solution of (10) in the form

$$\sum_{j=0}^{m} {}_1\gamma_j \left({}_{m-j}\mathfrak{m}_1, {}_{m-j}\mathfrak{m}_2, \ldots, {}_{m-j}\mathfrak{m}_n\right) = \hbar^{\zeta} \sum_{j=0}^{m} {}_2\gamma_j \mathrm{W}\left(\eta_{m-j}, {}_{m-j}\mathfrak{m}_1, {}_{m-j}\mathfrak{m}_2, \ldots, {}_{m-j}\mathfrak{m}_n\right), \tag{20}$$

for ${}_1\gamma_j, {}_2\gamma_j \in \mathbb{R}$. We will indicate with ${}_1\rho_m(\xi)$ and ${}_2\rho_m(\xi)$ the generating polynomials ${}_{\mathfrak{m}}\rho_m(\xi) = \Sigma_{j=0}^{m} {}_{\mathfrak{m}}\gamma_j \xi^{m-j}$. Numerical methods [28, 46] are requested to be consistent with the original problem (10), in the sense that, as $\hbar \to 0$, the discretized problem is expected to tend asymptotically to the continuous one [46]. The linear difference operator

$$\begin{aligned}\mathscr{L}_{\hbar}\Big[\left(\mathfrak{m}_1(v), \mathfrak{m}_2(v), \ldots, \mathfrak{m}_n(v)\right), v, \zeta\Big] &= \sum_{j=0}^{m} {}_1\gamma_j \Big({}_{m-j}\mathfrak{m}_1(v - \hbar j), {}_{m-j}\mathfrak{m}_2(v - \hbar j), \ldots, {}_{m-j}\mathfrak{m}_n(v - \hbar j)\Big) \\ &\quad - \hbar^{\zeta} \sum_{j=0}^{m} {}_2\gamma_j \mathbb{D}_q^{\zeta}\Big[{}_{m-j}\mathfrak{m}_1, {}_{m-j}\mathfrak{m}_2, \ldots, {}_{m-j}\mathfrak{m}_n\Big](v - \hbar j),\end{aligned}$$

where $(\mathfrak{m}_1(v), \mathfrak{m}_2(v), \ldots, \mathfrak{m}_n(v))$ is a sufficiently smooth function [46]. The linear multi-step method (20) is said to be consistent if, for any $\mathbb{F}q$-$\mathbb{DE}$ (10), with exact solution $(\mathfrak{m}_1(v), \mathfrak{m}_2(v), \ldots, \mathfrak{m}_n(v))$, it holds

$$\lim_{\hbar\to 0} \frac{1}{\hbar^{\zeta}} \mathscr{L}_{\hbar}\Big[\left({}_{m-j}\mathfrak{m}_1(v), {}_{m-j}\mathfrak{m}_2(v), \ldots, {}_{m-j}\mathfrak{m}_n(v)\right), v, \zeta\Big] = (0, 0, \ldots, 0),$$

Under the assumption that $(\mathfrak{m}_1(v), \mathfrak{m}_2(v), \ldots, \mathfrak{m}_n(v))$ is $(n+1)$–times differentiable, $v = \eta_m$, we can expand the true solution

$$\big(\mathfrak{m}_1(v - j\hbar), \mathfrak{m}_2(v - j\hbar), \ldots, \mathfrak{m}_n(v - j\hbar)\big) = \big(\mathfrak{m}_1(\eta_0 + (m-j)\hbar), \mathfrak{m}_2(\eta_0 + (m-j)\hbar), \ldots, \mathfrak{m}_n(\eta_0 + (m-j)\hbar)\big),$$

of (10) as

$$\big(\mathfrak{m}_1(v-j\hbar),\mathfrak{m}_2(v-j\hbar),\dots,\mathfrak{m}_n(v-j\hbar)\big)=\big(\mathfrak{m}_1(\eta_0),\mathfrak{m}_2(\eta_0),\dots,\mathfrak{m}_n(\eta_0)\big)$$
$$+\sum_{d=1}^{n}\tfrac{(m-j)^d\hbar^d}{d!}\left(\mathfrak{m}_1^d(\eta_0),\mathfrak{m}_2^d(\eta_0),\dots,\mathfrak{m}_n^d(\eta_0)\right)$$
$$+\tfrac{\hbar^{n+1}}{d!}\int_0^{m-j}(m-j-\xi)_q^{(n)}\Big(\mathfrak{m}_1^{n+1}(\eta_0+\hbar\xi),\ \mathfrak{m}_2^{n+1}(\eta_0+h\xi),\dots,\mathfrak{m}_n^{n+1}(\eta_0+\hbar\xi)\Big)\,\mathrm{d}_q\xi,$$

and its ζ-fractional q-derivative as

$$\mathbb{D}_q^{\zeta}\left[{}_{m-j}\mathfrak{m}_1,{}_{m-j}\mathfrak{m}_2,\dots,{}_{m-j}\mathfrak{m}_n\right](v-j\hbar)=\sum_{d=1}^{n}\tfrac{\hbar^{d-\zeta}(m-j)^{d-\zeta}}{\Gamma_q(d+1-\zeta)}\left(\mathfrak{m}_1^d(\eta_0),\mathfrak{m}_2^d(\eta_0),\dots,\mathfrak{m}_n^d(\eta_0)\right)$$
$$+\tfrac{\hbar^{n+1-\zeta}}{\Gamma_q(n+1-\zeta)}\int_0^{m-j}(m-j-\xi)_q^{(n-\zeta)}\left(\mathfrak{m}_1^{n+1}(\eta_0+\hbar\xi),\ \mathfrak{m}_2^{n+1}(\eta_0+\hbar\xi),\dots,\mathfrak{m}_4^{n+1}(\eta_0+\hbar\xi)\right)\mathrm{d}_q\xi.$$

So, we are able type in the difference operator $\mathscr{L}_\hbar\left[(\mathfrak{m}_1(v),\mathfrak{m}_2(v),\dots,\mathfrak{m}_n(v)),v,\zeta\right]$, as

$$\mathscr{L}_\hbar\Big[\big(\mathfrak{m}_1(v),\mathfrak{m}_2(v),\dots,\mathfrak{m}_n(v)\big),v,\zeta\Big]=C_0(m,\zeta)+\sum_{d=1}^{n}\hbar^dC_d(m,\zeta)\left(\mathfrak{m}_1^d(\eta_0),\mathfrak{m}_2^d(\eta_0),\dots,\mathfrak{m}_n^d(\eta_0)\right)+\hbar^{n+1}R_{n+1},$$

where the remainder R_{n+1} is obtained from the Taylor's expansions and for $d=0,1,2,\cdots,m$,

$$C_0(m,\tau)=\sum_{j=0}^{m}{}_1\gamma_j,\qquad C_d(m,\tau)=\frac{1}{d!}\sum_{j=0}^{m}(m-j)^d{}_1\gamma_j-\tfrac{1}{\Gamma_q(d+1-\zeta)}\sum_{j=0}^{m}{}_2\gamma_j(m-j)^{d-\zeta}.$$

3. Main results

We need the following key lemma.

Lemma 3.1. *Let $\sigma\in C(\overline{\Delta})$, the solution of the equation*

$$\mathbb{D}_q^{\zeta_0}\mathfrak{m}(v)=\sigma(v),\qquad v\in\overline{\Delta},\ n-1<\zeta_0<n,\tag{21}$$

subject to the boundary condition

$$\begin{cases}\mathfrak{m}(0)=\mathfrak{m}_\circ,\\ \mathfrak{m}'(0)=\mathfrak{m}''(0)=\cdots=\mathfrak{m}^{(n-2)}(0)=0,\\ \mathbb{I}_q^{\gamma}\mathfrak{m}(1)=\lambda\mathbb{I}_q^{\gamma}\mathfrak{m}(\xi),\end{cases}\tag{22}$$

is given by

$$\mathfrak{m}(v)=\tfrac{1}{\Gamma_q(\zeta_0)}\int_0^v(v-\eta)_q^{(\zeta_0-1)}\sigma(\eta)\,\mathrm{d}_q\eta+\mathfrak{m}_\circ-\tfrac{\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^1(1-\eta)_q^{(\zeta_0+\gamma-1)}\sigma(\eta)\,\mathrm{d}_q\eta$$
$$+\tfrac{\lambda\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^\xi(\xi-\eta)_q^{(\zeta_0+\gamma-1)}\sigma(\eta)\,\mathrm{d}_q\eta+\tfrac{\mathfrak{m}_\circ(\lambda\xi^{\gamma}-1)\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\gamma+1)}.\tag{23}$$

Proof. By Lemmas 2.4, 2.5, the general solution of (23) is given by

$$\mathfrak{m}(v) = \frac{1}{\Gamma_q(\zeta_0)} \int_0^v (v-\eta)_q^{\zeta_0 - 1} \sigma(\eta)\, \mathrm{d}_q\eta - c_0 - c_1 v - c_2 v^2 - \ldots - c_{n-1} v^{n-1}. \tag{24}$$

So, $\mathfrak{m}(0) = \mathfrak{m}_\circ$ and $\mathfrak{m}'(0) = \ldots = \mathfrak{m}^{(n-2)}(0) = 0$ implies that $c_0 = -\mathfrak{m}_\circ$ and $c_1 = c_2 = \ldots = c_{n-2} = 0$. Thanks to Lemma 2.2, we get

$$\mathbb{I}_q^\gamma \mathfrak{m}(v) = \frac{1}{\Gamma_q(\zeta_0+\gamma)} \int_0^v (v-\eta)_q^{(\zeta_0+\gamma-1)} \sigma(v)\, \mathrm{d}_q\eta + \frac{\mathfrak{m}_\circ v^\gamma}{\Gamma_q(\gamma+1)} - c_{n-1} \frac{\Gamma_q(n)}{\Gamma_q(\gamma+n)} v^{\gamma+n-1}. \tag{25}$$

Using the boundary condition $\mathbb{I}_q^\gamma \mathfrak{m}(1) = \lambda \mathbb{I}_q^\gamma \mathfrak{m}(\xi)$, we get

$$\begin{aligned} c_{n-1} = {} & \frac{\Gamma_q(\gamma+n)}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)} \sigma(\eta)\, \mathrm{d}_q\eta \\ & - \frac{\lambda\Gamma_q(\gamma+n)}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)} \sigma(\eta)\, \mathrm{d}_q\eta - \frac{\mathfrak{m}_\circ(\lambda\xi^\gamma - 1)\Gamma_q(\gamma+n)}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\gamma+1)}. \end{aligned} \tag{26}$$

Substituting the values of $c_0, c_1, \ldots, c_{n-1}$ in (24), we obtain the desired quantity in Lemma. □

Theorem 3.1. *Suppose that* $\xi^{\gamma+n-1} \neq \frac{1}{\lambda}$, *and assume that the hypothesis* (P1) *holds:*

(P1)*: There exist non negative continuous functions* $\varkappa_i \in C(\overline{\Delta})$, $i = 0, 1, 2, \ldots, n-1$ *such that* $\forall v \in \overline{\Delta}$ *and* $(\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \ldots, \mathrm{m}_{n-1})$, $(\acute{\mathrm{m}}_0, \acute{\mathrm{m}}_1, \acute{\mathrm{m}}_2, \ldots, \acute{\mathrm{m}}_{n-1}) \in \mathbb{R}^n$, *we have*

$$|\mathrm{w}(v, \mathrm{m}_0, \mathrm{m}_1, \ldots, \mathrm{m}_{n-1}) - \mathrm{w}(v, \acute{\mathrm{m}}_0, \acute{\mathrm{m}}_1, \ldots, \acute{\mathrm{m}}_{n-1})| \leq \sum_{i=0}^{n-1} \varkappa_i(v) |\mathrm{m}_i - \acute{\mathrm{m}}_i|,$$

where, $\varkappa_i^* = \sup_{v \in \overline{\Delta}} \varkappa_i(v)$, $(i = 0, 1, 2, \ldots, n-1)$.

If $\mathring{\kappa} := \varkappa_0^* + \varkappa_1^* + \cdots + \varkappa_{n-1}^*$,

$$\Upsilon := \left(L_0 + \sum_{i=1}^{n-1} L_i \right) \mathring{\kappa} < 1, \tag{27}$$

where

$$\begin{aligned} L_0 &:= \frac{1}{\Gamma_q(\zeta_0+1)} + \frac{\Gamma_q(\gamma+n)(1+|\lambda|\xi^{\gamma+\zeta_0})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0+1)}, \\ L_i &:= \frac{1}{\Gamma_q(\zeta_0-\zeta_i+1)} + \frac{\Gamma_q(\gamma+n)(1+|\lambda|\xi^{\zeta_0+\gamma})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_i)\Gamma_q(\zeta_0+\gamma+1)}, \qquad i = 1, \ldots, n-1, \end{aligned} \tag{28}$$

then the q*-fractional problem* (10) *has a unique solution.*

Proof. Consider the operator $\mathcal{O} : X \to X$ defined by

$$\begin{aligned} \mathcal{O}\mathfrak{m}(v) = {} & \frac{1}{\Gamma_q(\zeta_0)} \int_0^v (v-\eta)_q^{(\zeta_0-1)} \tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\, \mathrm{d}_q\eta + \mathfrak{m}_\circ - \frac{\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)} \tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\, \mathrm{d}_q\eta \\ & + \frac{\lambda\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)} \tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\, \mathrm{d}_q\eta + \frac{\mathfrak{m}_\circ(\lambda\xi^\gamma - 1)\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\gamma+1)}. \end{aligned} \tag{29}$$

where $\tilde{\mathrm{w}}_{\mathfrak{m}}(v) = \mathrm{w}\big(v, \mathfrak{m}(v), \mathbb{D}_q^{\zeta_1}\mathfrak{m}(v), \dots, \mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m}(v)\big)$. We shall prove that $\mathcal{O}$ is contraction mapping. For $\mathfrak{m}, \acute{\mathfrak{m}} \in X$ and for each $v \in \overline{\Delta}$, we have

$$
\begin{aligned}
|\mathcal{O}\mathfrak{m}(v) - \mathcal{O}\acute{\mathfrak{m}}(v)| \leq{}& \tfrac{1}{\Gamma_q(\zeta_0)} \int_0^v (v-\eta)_q^{(\zeta_0-1)} \left|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta) - \tilde{\mathrm{w}}_{\acute{\mathfrak{m}}}(\eta)\right| \,\mathrm{d}_q\eta \\
&+ \tfrac{\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)} \left|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta) - \tilde{\mathrm{w}}_{\acute{\mathfrak{m}}}(\eta)\right| \,\mathrm{d}_q\eta \\
&+ \tfrac{|\lambda|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)} \left|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta) - \tilde{\mathrm{w}}_{\acute{\mathfrak{m}}}(\eta)\right| \,\mathrm{d}_q\eta. \qquad (30)
\end{aligned}
$$

Using the (P1), we can write

$$
\begin{aligned}
|\mathcal{O}\mathfrak{m}(v) - \mathcal{O}\acute{\mathfrak{m}}(v)| \leq{}& \tfrac{\mathring{\kappa}}{\Gamma_q(\zeta_0+1)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big] \\
&+ \tfrac{\mathring{\kappa}\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big] \\
&+ \tfrac{|\lambda|\Gamma_q(\gamma+n)\xi^{\zeta_0+\gamma}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma+1)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big]. \qquad (31)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\mathcal{O}\mathfrak{m}(v) - \mathcal{O}\acute{\mathfrak{m}}(v)| \leq{}& \tfrac{\mathring{\kappa}}{\Gamma_q(\zeta_0+1)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big] \\
&+ \tfrac{\mathring{\kappa}\Gamma_q(\gamma+n)\left[1-|\lambda|\xi^{\gamma+\zeta_0}\right]}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big]. \qquad (32)
\end{aligned}
$$

Consequently, we have

$$
|\mathcal{O}\mathfrak{m}(v) - \mathcal{O}\acute{\mathfrak{m}}(v)| \leq L_0\mathring{\kappa}\Big(\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big), \qquad (33)
$$

which implies that

$$
\|\mathcal{O}(\mathfrak{m}) - \mathcal{O}(\acute{\mathfrak{m}})\| \leq L_0\mathring{\kappa}\Big(\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big). \qquad (34)
$$

Also, $\forall\, k = 1, \dots, n-1$ and for each $v \in \overline{\Delta}$, we have

$$
\begin{aligned}
\left|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}(v) - \mathbb{D}_q^{\zeta_k}\mathcal{O}\acute{\mathfrak{m}}(v)\right| \leq{}& \tfrac{1}{\Gamma_q(\zeta_0-\zeta_k)} \int_0^v (v-\eta)_q^{(\zeta_0-\zeta_k-1)} \left|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta) - \tilde{\mathrm{w}}_{\acute{\mathfrak{m}}}(\eta)\right| \,\mathrm{d}_q\eta \\
&+ \tfrac{\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)} \left|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta) - \tilde{\mathrm{w}}_{\acute{\mathfrak{m}}}(\eta)\right| \,\mathrm{d}_q\eta \\
&+ \tfrac{|\lambda|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)} \left|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta) - \tilde{\mathrm{w}}_{\acute{\mathfrak{m}}}(\eta)\right| \,\mathrm{d}_q\eta. \qquad (35)
\end{aligned}
$$

By (P1) we obtain

$$
\begin{aligned}
\left|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}(v) - \mathbb{D}_q^{\zeta_k}\mathcal{O}\acute{\mathfrak{m}}(v)\right| \leq{}& \tfrac{\mathring{\kappa}}{\Gamma_q(\zeta_0-\zeta_k+1)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big] \\
&+ \tfrac{\mathring{\kappa}\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma)} \Big[\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \dots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big]
\end{aligned}
$$

$$+ \tfrac{|\lambda|\Gamma_q(\gamma+n)\xi^{\zeta_0+\gamma}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)}\Big[\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big]. \quad (36)$$

Hence,

$$\Big|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}(v) - \mathbb{D}_q^{\zeta_k}\mathcal{O}\acute{\mathfrak{m}}(v)\Big| \leq \tfrac{\mathring{\kappa}}{\Gamma_q(\zeta_0-\zeta_k+1)}\Big[\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big]$$
$$+ \tfrac{|\lambda|\Gamma_q(\gamma+n)\xi^{\zeta_0+\gamma}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)}\Big[\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big]. \quad (37)$$

Thus,

$$\Big|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}(v) - \mathbb{D}_q^{\zeta_k}\mathcal{O}\acute{\mathfrak{m}}(v)| \leq L_k\mathring{\kappa}\Big(\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big), \quad (38)$$

which implies that

$$\Big\|\mathbb{D}_q^{\zeta_k}\mathcal{O}(\mathfrak{m}) - \mathbb{D}_q^{\zeta_k}\mathcal{O}(\acute{\mathfrak{m}})\Big\| \leq L_k\mathring{\kappa}\Big(\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\Big). \quad (39)$$

Thanks to (34) and (39), we obtain

$$\|\mathcal{O}(\mathfrak{m}) - \mathcal{O}(\acute{\mathfrak{m}})\|_X \leq \Big(L_0 + \sum_{k=0}^{n-1} L_k\Big)\mathring{\kappa}\Big(\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\gamma-1}\mathfrak{m} - \mathbb{D}_q^{\gamma-1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\gamma_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\gamma_{n-1}}\acute{\mathfrak{m}}\|\Big), \quad (40)$$

which is contraction. As a consequence of Banach contraction, we deduce that $\mathcal{O}$ has a fixed point which is a solution of the boundary value Problem (10). □

Theorem 3.2. *Suppose that for all $\xi^{\gamma+n-1} \neq \frac{1}{\lambda}$ and assume that the following hypotheses (H1) and (H3) are satisfied:*

(P2) *The function $\mathrm{w} : \overline{\Delta} \times \mathbb{R}^n \to \mathbb{R}$ is continuous;*

(P3) *There exist positive constants $\varrho(v)$ such that*

$$|\mathrm{w}(v, \mathrm{m}_0, \mathrm{m}_1, \dots, \mathrm{m}_{n-1})| \leq \varrho(v), \qquad \forall\, v \in \overline{\Delta}\, \&\, \forall\, (\mathrm{m}_0, \mathrm{m}_1, \dots, \mathrm{m}_{n-1}) \in \mathbb{R}^n,$$

with $\varrho^* = \sup_{v\in\overline{\Delta}} \varrho(v)$.

Then, the problem (10) has at least a solution on $\overline{\Delta}$.

Proof. We shall use Scheafer's fixed point theorem to prove that $\mathcal{O}$ has at least a fixed point on X. By (P2) we conclude that the operator $\mathcal{O}$ is continuous. For $r_\circ > 0$, we take $v \in \mathrm{B}_{r_\circ} = \big\{v \in X : \|v\|_X \leq r_\circ\big\}$. For each $v \in \overline{\Delta}$, we have

$$|\mathcal{O}\mathfrak{m}(v)| = \tfrac{1}{\Gamma_q(\zeta_0)}\int_0^v (v-\eta)_q^{(\zeta_0-1)}|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta + |\mathfrak{m}_\circ|$$
$$+ \tfrac{\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)}|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta$$
$$+ \tfrac{|\lambda|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)}|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta + \tfrac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \quad (41)$$

Using the (P3), we obtain

$$|\mathcal{O}\mathfrak{m}(v)| \leq \sup_{v\in J}\varrho(v)\left[\frac{1}{\Gamma_q(\zeta_0+1)} + \frac{\Gamma_q(\gamma+n)(1+|\lambda|\eta^{\gamma+\zeta_0})}{|1-\lambda\eta^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0)}\right] + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ||\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{42}$$

Thus,

$$|\mathcal{O}\mathfrak{m}(v)| \leq L_0 \sup_{v\in J}\varrho(v) + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{43}$$

It follows that

$$\|\mathcal{O}\mathfrak{m}(v)\| \leq L_0\varrho^* + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{44}$$

On the other hand, $\forall\, k = 1, 2, \dots, n-1$, we get

$$\begin{aligned}\left|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}(v)\right| &= \int_0^v \frac{(v-\eta)_q^{(\zeta_0-\zeta_k-1)}}{\Gamma_q(\zeta_0-\zeta_k)}\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\,\mathrm{d}_q\eta + \mathfrak{m}_\circ \\ &\quad - \frac{\Gamma_q(\gamma+n)v^{n-\zeta_k-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma)}\int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)}\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\,\mathrm{d}_q\eta \\ &\quad + \frac{|\lambda|\,\Gamma_q(\gamma+n)v^{n-\zeta_k-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)}\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\,\mathrm{d}_q\eta + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)v^{n-\zeta_k-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\gamma+1)}.\end{aligned} \tag{45}$$

Again by (P3), we have

$$\left\|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}\right\| \leq \varrho^*\left[\frac{1}{\Gamma_q(\zeta_0-\zeta_k+1)} + \frac{\Gamma_q(\gamma+n)\big(1+L_k|\lambda|\eta^{\gamma+\zeta_0}\big)}{|1-\lambda\eta^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+\zeta_0+1)}\right] + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}. \tag{46}$$

Hence

$$\left\|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathfrak{m}\right\| \leq \varrho^* L_k + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}, \tag{47}$$

here $k = 1, 2, \dots, n-1$. Combining Eqs. (44) and (47), yields

$$\|\mathcal{O}(\mathfrak{m})\|_X \leq \varrho^*\left(L_0 + \sum_{k=0}^{n-1} L_k\right) + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)} + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}, \tag{48}$$

Indeed, $\|\mathcal{O}(\mathfrak{m})\|_X \leq \infty$. Let us take $(\mathfrak{m}, \acute{\mathfrak{m}}) \in \mathrm{B}_{r_\circ}$, $v_1, v_2 \in \overline{\Delta}$, such that $v_1 < v_2$ and thanks to (P3). We can write

$$\begin{aligned}|\mathcal{O}\mathfrak{m}(v_2) - \mathcal{O}\mathfrak{m}(v_1)| &\leq \sup_{v\in\overline{\Delta}}\varrho(v)\int_0^{v_1}\frac{(v_2-\eta)^{\zeta_0-1}-(v_1-\eta)^{\zeta_0-1}}{\Gamma_q(\zeta_0)}\,\mathrm{d}_q\eta + \sup_{v\in\overline{\Delta}}\varrho(v)\int_{v_1}^{v_2}\frac{(v_2-\eta)^{\zeta_0-1}}{\Gamma_q(\zeta_0)}\,\mathrm{d}_q\eta \\ &\quad + \sup_{v\in\overline{\Delta}}\varrho(v)\frac{\Gamma_q(\gamma+n)(v_1^{n-1}-v_2^{n-1})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0)}\int_0^1 (1-\eta)^{\gamma+\zeta_0-1}\,\mathrm{d}_q\eta \\ &\quad + \sup_{v\in\overline{\Delta}}\varrho(v)\frac{|\lambda|\Gamma_q(\gamma+n)(v_2^{n-1}-v_1^{n-1})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0)}\int_0^\xi (\xi-\eta)^{\gamma+\zeta_0-1}\,\mathrm{d}_q\eta + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)(v_2^{n-1}-v_1^{n-1})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}.\end{aligned} \tag{49}$$

Thus,

$$|\mathcal{O}\mathfrak{m}(v_2) - \mathcal{O}\mathfrak{m}(v_1)| \leq \frac{\varrho^*}{\Gamma_q(\zeta_0+1)}\left(v_1^{\zeta_0} - v_2^{\zeta_0}\right) + \frac{2\varrho^*}{\Gamma_q(\zeta_0+1)}\left(v_2 - v_1\right)^{\zeta_0}$$

$$+ \frac{\varrho^* \Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0+1)} \left(v_1^{n-1} - v_2^{n-1}\right)$$
$$+ \left[\frac{\varrho^* |\lambda| \Gamma_q(\gamma+n)\xi^{\gamma+\zeta_0}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0+1)} + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}\right] \left(v_2^{n-1} - v_1^{n-1}\right). \tag{50}$$

Also, $\forall\, k = 1, 2 \dots, n-1$,

$$|\mathbb{D}_q^{\zeta}\mathcal{O}_1\acute{\mathrm{m}}(v_2) - \mathbb{D}_q^{\zeta}\mathcal{O}_1\acute{\mathrm{m}}(v_1)| \le \frac{\varrho^*}{\Gamma_q(\zeta_0-\zeta_k+1)} \left(v_1^{\zeta_0-\zeta_k} - v_2^{\zeta_0-\zeta_k}\right) + \frac{2\varrho^*}{\Gamma_q(\zeta_0-\zeta_k+1)}(v_2-v_1)^{\zeta_0-\zeta_k}$$
$$+ \frac{\varrho^* \Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+\zeta_0+1)} \left(v_1^{n-\zeta_k-1} - v_2^{n-\zeta_k-1}\right)$$
$$+ \frac{\varrho^* |\lambda|\Gamma_q(\gamma+n)\xi^{\gamma+\zeta_0}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+\zeta_0+1)} \left(v_2^{n-\zeta_k-1} - v_1^{n-\zeta_k-1}\right)$$
$$+ \frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)} \left(v_2^{n-\zeta_k-1} - v_1^{n-\zeta_k-1}\right). \tag{51}$$

Thanks to Eqs. (50) and (51), we can state that $\|\mathcal{O}(\mathrm{m})(v_2) - \mathcal{O}(\mathrm{m})(v_1)\| \to 0$, as $v_2 \to v_1$. By Arzelá-Ascoli theorem, we conclude that $\mathcal{O}$ is completely continuous operator. Finally, we discuss a priory bounds on solutions. We shall show that the set Ω defined by

$$\Omega = \Big\{\mathrm{m} \in X \,:\, \mathrm{m} = r_\circ\mathcal{O}(\mathrm{m}),\, 0 < r_\circ < 1\Big\}, \tag{52}$$

is bounded. Let $\mathrm{m} \in \Omega$, then $\mathrm{m} = r_\circ\mathcal{O}(\mathrm{m})$, for some $0 < r_\circ < 1$. Thus, for each $v \in \overline{\Delta}$, we have

$$\mathrm{m}(v) = r_\circ\mathcal{O}(\mathrm{m}), \tag{53}$$

then

$$\frac{1}{r_\circ}|\mathfrak{m}(v)| = \frac{1}{\Gamma_q(\zeta_0)} \int_0^v (v-\eta)_q^{(\zeta_0-1)} |\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)| \,\mathrm{d}_q\eta + |\mathfrak{m}_\circ|$$
$$+ \frac{\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)} |\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)| \,\mathrm{d}_q\eta$$
$$+ \frac{|\lambda|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)} |\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)| \,\mathrm{d}_q\eta + \frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{54}$$

Condition (P3) implies that

$$\frac{1}{r_\circ}|\mathrm{m}(v)| \le \sup_{v\in\overline{\Delta}} \varrho(v) \left[\frac{1}{\Gamma_q(\zeta_0+1)} + \frac{\Gamma_q(\gamma+n)(1+|\lambda|\eta^{\gamma+\zeta_0})}{|1-\lambda\eta^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0)}\right] + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{55}$$

Therefore,

$$|\mathrm{m}(v)| \le r_\circ \sup_{v\in J} \varrho(t) \left[\frac{1}{\Gamma_q(\zeta_0+1)} + \frac{\Gamma_q(\gamma+n)(1+|\lambda|\xi^{\gamma+\zeta_0})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0)}\right] + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{56}$$

Hence,

$$|\mathrm{m}(v)| \le r_\circ L_0 \sup_{v\in J} \varrho(v) + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}, \tag{57}$$

$\forall\, v \in \overline{\Delta}$, which implies that,

$$\|\mathrm{m}\| \le r_\circ\varrho^* L_0 + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{58}$$

Similarly, $\forall\, k = 1, 2, \dots, n-1$, we have

$$|\mathbb{D}_q^{\zeta_k}\mathcal{O}\mathrm{m}(v)| \le r_\circ \sup_{v\in\overline{\Delta}} \varrho(v)\left[\frac{1}{\Gamma_q(\zeta_0-\zeta_k+1)} + \frac{\Gamma_q(\gamma+n)(1+|\lambda|\xi^{\gamma+\zeta_0})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+\zeta_0+1)}\right] + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}, \tag{59}$$

for all $v \in \overline{\Delta}$. Thus,

$$\|\mathbb{D}_q^{\zeta_k}\mathcal{O}(\mathrm{m})\| \le r_\circ \varrho^* L_k + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}. \tag{60}$$

From (58) and (60) we get,

$$\|\mathrm{m}\|_X \le r_\circ\varrho^*\left(L_0 + \sum_{k=1}^{n-1} L_k\right) + |\mathfrak{m}_\circ| + \frac{|\mathfrak{m}_\circ||\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)[\Gamma_q(n-\zeta_k)+\Gamma_q(n)]}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(n-\zeta_k)\Gamma(\gamma+1)}. \tag{61}$$

Hence $\|\mathcal{O}(\mathrm{m})\|_X < \infty$. This shows that the set Ω is bounded. As consequence of Schaefer's fixed point theorem, we deduce that $\wr$ at least a fixed point, which is a solution of the $\mathbb{FDE}$ (10). □

Our third main result is based on Krasnoselskii Theorem in [48].

Theorem 3.3. *Let $\xi^{\gamma+n-1} \neq \frac{1}{\lambda}$, and the hypothesis* (P1), (P2) *and* (P3) *are satisfied, such that*

$$\Upsilon_1 := \left(\frac{1}{\Gamma_q(\zeta_0+1)} + \sum_{i=1}^{n-1}\frac{1}{\Gamma_q(\zeta_0-\zeta_k+1)}\right)\mathring{\kappa} < 1. \tag{62}$$

If there exist $\hat{r} \in \mathbb{R}^+$ such that

$$\Upsilon_2 := \varrho^*\left(L_0 + \sum_{i=1}^{n-1} L_i\right) + |\mathfrak{m}_\circ| + \sum_{i=1}^{n-1}\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)\left[\Gamma_q(n-\zeta_k)+\Gamma_q(n)\right]}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(n)\Gamma_q(\gamma+1)} < \hat{r}. \tag{63}$$

Then the $\mathbb{F}q$-$\mathbb{DE}$ has at leat one solution on $\overline{\Delta}$.

Proof. We shall use Krasnoselskii fixed point theorem to prove that ϕ has at least a fixed point on $\mathcal{X}$. Suppose that inequality (63) holds and let us take $\mathcal{O}\mathfrak{m}(v) := \top_1\mathfrak{m}(v) + \top_2\mathfrak{m}(v)$ where

$$\top_1\mathfrak{m}(v) := \frac{1}{\Gamma_q(\zeta_0)}\int_0^v (v-\eta)_q^{(\zeta_0-1)}\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\,\mathrm{d}_q\eta + \mathfrak{m}_\circ, \tag{64}$$

$$\begin{aligned}\top_2\mathfrak{m}(v) := &-\frac{\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)}\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\,\mathrm{d}_q\eta \\ &+ \frac{\lambda\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^\xi (\xi-\eta)_q^{(\zeta_0+\gamma-1)}\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)\,\mathrm{d}_q\eta + \frac{\mathfrak{m}_\circ(\lambda\xi^{\gamma}-1)\Gamma_q(\gamma+n)v^{n-1}}{(1-\lambda\xi^{\gamma+n-1})\Gamma_q(n)\Gamma_q(\gamma+1)}.\end{aligned} \tag{65}$$

The proof will be given in several steps. We shall prove that for any $\mathfrak{m}, \acute{\mathfrak{m}} \in B_\delta$, then $\top_1\mathfrak{m}(v) + \top_2\acute{\mathfrak{m}}(v) \in r_\circ$, such that $\mathrm{B}_{r_\circ} = \{v \in \mathcal{X} : \|v\|_{\mathcal{X}} \le r_\circ\}$. For each $\mathfrak{m}, \acute{\mathfrak{m}} \in B_\delta$ and $v \in \overline{\Delta}$, we have

$$|\top_1\mathfrak{m}(v) + \top_2\acute{\mathfrak{m}}(v)| = \frac{1}{\Gamma_q(\zeta_0)}\int_0^v (v-\eta)_q^{(\zeta_0-1)}|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta + |\mathfrak{m}_\circ| \tag{66}$$

$$+\frac{\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^1(1-\eta)_q^{(\zeta_0+\gamma-1)}|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta$$

$$+\frac{|\lambda|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^\xi(\xi-\eta)_q^{(\zeta_0+\gamma-1)}|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta+\frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}.$$

Using the (P3), we obtain

$$|\top_1\mathfrak{m}(v)+\top_2\acute{\mathfrak{m}}(v)|\le\sup_{v\in J}\varrho(v)\left[\frac{1}{\Gamma_q(\zeta_0+1)}+\frac{\Gamma_q(\gamma+n)(1+|\lambda|\eta^{\gamma+\zeta_0})}{|1-\lambda\eta^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+\zeta_0)}\right]+|\mathfrak{m}_\circ|+\frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{67}$$

Consequently

$$|\top_1\mathfrak{m}(v)+\top_2\acute{\mathfrak{m}}(v)|\le L_0\sup_{v\in J}\varrho(v)+|\mathfrak{m}_\circ|+\frac{|\mathfrak{m}_\circ||\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{68}$$

Thus,

$$\|\top_1\mathfrak{m}(v)+\top_2\acute{\mathfrak{m}}(v)\|\le L_0\varrho^*+|\mathfrak{m}_\circ|+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}, \tag{69}$$

and

$$\begin{aligned}\left|\mathbb{D}_q^{\zeta_k}\top_1\mathfrak{m}(v)-\mathbb{D}_q^{\zeta_k}\top_2\acute{\mathfrak{m}}(v)\right|&\le\sup_{v\in J}\varrho(v)\left[\frac{1}{\Gamma_q(\zeta_0-\zeta_k+1)}+\frac{\Gamma_q(\gamma+n)\big(1+L_k|\lambda|\eta^{\gamma+\zeta_0}\big)}{|1-\lambda\eta^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+\zeta_0+1)}\right]\\&\quad+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}\le\varrho^*L_k+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)},\end{aligned} \tag{70}$$

$\forall\, k=1,2,\dots,n-1$. Combining (69) and (70) yields

$$\|\top_1\mathfrak{m}+\top_2\acute{\mathfrak{m}}\|_{\mathcal{X}}\le\varrho^*\left(L_0+\sum_{i=1}^{n-1}L_i\right)+|\mathfrak{m}_\circ|+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)[\Gamma_q(n-\zeta_k)+\Gamma_q(n)]}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}. \tag{71}$$

Indeed $\top_1\mathfrak{m}+\top_2\acute{\mathfrak{m}}\in\mathrm{B}_{r_\circ}$. We shall prove that $\top_2$ is continuous and compact. Note that $\top_2$ is continuous on $\mathcal{X}$ in view of the continuity of w (by hypothesis (P2)). Now, we prove that $\top_2$ maps bounded sets into bounded sets of $\mathcal{X}$. For $\mathfrak{m}\in\mathrm{B}_{r_\circ}$ and for each $v\in\overline{\Delta}$, we have

$$\begin{aligned}|\top_2\mathfrak{m}(v)|&\le\frac{\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^1(1-\eta)_q^{(\zeta_0+\gamma-1)}\,|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta\\&\quad+\frac{|\lambda|\,\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)}\int_0^\xi(\xi-\eta)_q^{(\zeta_0+\gamma-1)}\,|\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\,\mathrm{d}_q\eta+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}.\end{aligned} \tag{72}$$

Using the (P3) we obtain

$$|\top_2\mathfrak{m}(v)|\le\sup_{v\in J}\varrho(v)\frac{\Gamma_q(\gamma+n)\big(1+\lambda\xi^{\gamma+\zeta_k}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma+1)}+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}, \tag{73}$$

$\forall\, v\in\overline{\Delta}$. Thus,

$$|\top_2\mathfrak{m}(v)|\le\frac{\varrho^*\Gamma_q(\gamma+n)\big(1+\lambda\xi^{\gamma+\zeta_k}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma+1)}+\frac{|\mathfrak{m}_\circ|\,|\lambda\xi^\gamma-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}. \tag{74}$$

On the other hand, for all $k = 1, 2, n-1$, we have

$$\|\mathbb{D}_q^{\zeta_k} \top_2 \mathfrak{m}(v)\| \leq \frac{\varrho^* \Gamma_q(\gamma+n)\big(1+\lambda\xi^{\gamma+\zeta_k}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)} + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)v^{n-1}}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}. \tag{75}$$

From (74) and (75), we have

$$\|\top_2 \mathfrak{m}(v)\|_{\mathcal{X}} \leq \varrho^* \left(\frac{\Gamma_q(\gamma+n)\big(1+\lambda\xi^{\gamma+\zeta_k}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)} + \sum_{k=1}^{n-1} \frac{\Gamma_q(\gamma+n)\big(1+\lambda\xi^{\gamma+\zeta_k}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+\zeta_0+1)} \right)$$
$$+ \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)} + \sum_{k=1}^{n-1} \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}.$$

Consequently, $\|\top_2 \mathfrak{m}(v)\|_{\mathcal{X}} \leq \varrho^* \Sigma_1 + \Sigma_2 < \infty$, where

$$\Sigma_1 := \frac{\Gamma_q(\gamma+n)(1+|\lambda|\xi^{\gamma+\zeta_k})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma+1)} + \sum_{k=1}^{n-1} \frac{\Gamma_q(\gamma+n)(1+|\lambda|\xi^{\zeta_k+\gamma})}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)},$$

$$\Sigma_2 := \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)} + \sum_{k=1}^{n-1} \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)}. \tag{76}$$

In the end we show that $\top_2$ is equicontinuous on $\overline{\Delta}$. Let $v_1, v_2 \in \overline{\Delta}$, such that $v_2 < v_1$ and $(\mathfrak{m},\ \acute{\mathfrak{m}}) \in \mathrm{B}_{r_\circ}$. Then, we have

$$|\top_2 \mathfrak{m}(v_1) - \top_2 \mathfrak{m}(v_2)| \leq \frac{\Gamma_q(\gamma+n)\big(v_2^{n-1}-v_1^{n-1}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^1 (1-\eta)_q^{(\zeta_0+\gamma-1)}\, |\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\, \mathrm{d}_q\eta \tag{77}$$
$$+ \frac{|\lambda|\,\Gamma_q(\gamma+n)\big(v_2^{n-1}-v_1^{n-1}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \int_0^{\xi} (\xi-\eta)_q^{(\zeta_0+\gamma-1)}\, |\tilde{\mathrm{w}}_{\mathfrak{m}}(\eta)|\, \mathrm{d}_q\eta + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)\big(v_2^{n-1}-v_1^{n-1}\big)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)}.$$

Using the (H3), we obtain

$$|\top_2 \mathfrak{m}(v_1) - \top_2 \mathfrak{m}(v_2)| \leq \frac{\varrho^*\,\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \left(v_2^{n-1} - v_1^{n-1}\right) \tag{78}$$
$$+ \frac{\varrho^*\,|\lambda|\,\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\zeta_0+\gamma)} \left(v_2^{n-1} - v_1^{n-1}\right) + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n)\Gamma_q(\gamma+1)} \left(v_2^{n-1} - v_1^{n-1}\right).$$

On the other hand, for all $k = 1, 2n-1$,

$$|\mathbb{D}_q^{\zeta_k} \top_2 \mathfrak{m}(v_1) - \mathbb{D}_q^{\zeta_k} \top_2 \mathfrak{m}(v_2)| \leq \frac{\varrho^*\,\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)} \left(v_2^{n-1} - v_1^{n-1}\right) \tag{79}$$
$$+ \frac{\varrho^*\,|\lambda|\,\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\zeta_0+\gamma+1)} \left(v_2^{n-1} - v_1^{n-1}\right) + \frac{|\mathfrak{m}_\circ|\,|\lambda\xi^{\gamma}-1|\Gamma_q(\gamma+n)}{|1-\lambda\xi^{\gamma+n-1}|\Gamma_q(n-\zeta_k)\Gamma_q(\gamma+1)} \left(v_2^{n-1} - v_1^{n-1}\right).$$

As $v_2 \to v_1$, the right-hand sides of the inequalities (78) and (79) tend to zero. Then, as a consequence of the above steps by Arzelá-Ascoli theorem, we conclude that $\top_2$ is completely continuous. Finally, we prove that $\top_1$ is contraction mapping. Let $\mathfrak{m}, \acute{\mathfrak{m}} \in \mathcal{X}$. Then, for each $v \in \overline{\Delta}$ and by (P1) we have

$$|\top_1 \mathfrak{m}(v) - \top_1 \acute{\mathfrak{m}}(v)| \leq \frac{\mathring{\kappa}}{\Gamma_q(\zeta_0+1)} \Big(\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\| \Big). \tag{80}$$

Also, for all $k = 1, 2, \ldots, n-1$, we have

$$|\mathbb{D}_q^{\zeta_k} \top_1 \mathfrak{m}(v) - \mathbb{D}_q^{\zeta_k} \top_1 \acute{\mathfrak{m}}(v)| \leq \frac{\mathring{\kappa}}{\Gamma_q(\zeta_0-\zeta_k+1)} \Big(\|\mathfrak{m} - \acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\| \Big). \tag{81}$$

By (80) and (81), we obtain

$$\|\top_1\mathfrak{m}(v) - \top_1\acute{\mathfrak{m}}(v)\| \leq \left[\frac{1}{\Gamma_q(\zeta_0+1)} + \sum_{k=1}^{n-1}\frac{1}{\Gamma_q(\zeta_0-\zeta_k+1)}\right]\mathring{\kappa}$$
$$\times\left[\|\mathfrak{m}-\acute{\mathfrak{m}}\| + \|\mathbb{D}_q^{\zeta_1}\mathfrak{m} - \mathbb{D}_q^{\zeta_1}\acute{\mathfrak{m}}\| + \cdots + \|\mathbb{D}_q^{\zeta_{n-1}}\mathfrak{m} - \mathbb{D}_q^{\zeta_{n-1}}\acute{\mathfrak{m}}\|\right]. \tag{82}$$

Indeed $\top_1$ is a contraction mapping. As a consequence of Krasnoselskii's fixed point theorem we deduce that $\mathcal{O}$ has a fixed point which is a solution of (10). □

Corollary 3.1. *Assume that* $\xi^{\gamma+n-1} \neq \frac{1}{\lambda}$, *and there exist non negative real numbers* ϑ_i, $i = 0, 1, n-1$ *such that* $\forall v \in \overline{\Delta}$ *and* $(\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_{n-1})$, $(\acute{\mathrm{m}}_0, \acute{\mathrm{m}}_1, \acute{\mathrm{m}}_2, \acute{\mathrm{m}}_{n-1}) \in \mathbb{R}^n$, *we have*

$$|\mathrm{w}(v, \mathrm{m}_0, \mathrm{m}_1, \dots, \mathrm{m}_{n-1}) - \mathrm{w}(v, \acute{\mathrm{m}}_0, \acute{\mathrm{m}}_1, \dots, \acute{\mathrm{m}}_{n-1})| \leq \sum_{i=0}^{n-1}\vartheta_i|\mathrm{m}_i - \acute{\mathrm{m}}_i|,$$

if

$$\left(L_0 + \sum_{i=1}^{n-1} L_i\right)(\vartheta_0 + \vartheta_1 + \cdots + \vartheta_{n-1}) < 1, \tag{83}$$

then the fractional problem (10) *has a unique solution on* $\overline{\Delta}$.

Corollary 3.2. *Assume that* (P2) *holds and* $\xi^{\gamma+n-1} \neq \frac{1}{\lambda}$. *If there exist positive constants* k_1 *and* k_2 *such that* $\mathrm{w}_1 \leq k_1, \mathrm{w}_2 \leq k_2$ *on* $\overline{\Delta} \times \mathbb{R}^n$, *then, the problem* (10) *has at least a solution on* $\overline{\Delta}$.

4. Illustrative examples under numerical algorithms

In the following, we give a few examples which show our all results hold.

Example 4.1. *Consider the following* $\mathbb{F}q$-$\mathbb{DE}$ *with the boundary condition,* $q \in \{0.13,\ 0.5,\ 0.89\}$,

$$\begin{cases} \mathbb{D}_q^{15/4}\mathfrak{m}(v) = \frac{|\mathfrak{m}(v)| + \left|\mathbb{D}_q^{11/5}\mathfrak{m}(v)\right| + \left|\mathbb{D}_q^{5/3}\mathfrak{m}(v)\right| + \left|\mathbb{D}_q^{6/7}\mathfrak{m}(v)\right|}{\left(v^2+35\pi\right)\left(e^v + |x(t)| + \left|\mathbb{D}_q^{11/5}\mathfrak{m}(v)\right| + \left|\mathbb{D}_q^{5/3}\mathfrak{m}(v)\right| + \left|\mathbb{D}_q^{6/7}\mathfrak{m}(v)\right|\right)} + \cosh(2+v^2), \\ \begin{cases} \mathfrak{m}(0) = \sqrt{3}, \\ \mathfrak{m}'(0) = \mathfrak{m}''(0) = 0, \\ \mathbb{I}_q^{1/2}\mathfrak{m}(1) = 0.81\,\mathbb{I}_q^{1/2}\mathfrak{m}(0.72), \end{cases} \end{cases} \tag{84}$$

for $v \in \overline{\Delta}$. *It goes without saying* $\zeta_0 = \frac{15}{4} < n = 4$, $\zeta_1 = \frac{11}{5} < \zeta_0$, $\zeta_2 = \frac{5}{3} < \zeta_1$, $\zeta_3 = \frac{6}{7} < \zeta_2$, $\mathfrak{m}_\circ = \sqrt{3}$, $\gamma = \frac{1}{2} \in \Delta$, $\lambda = 0.81 \neq 0$, $\xi = 0.72 \in \Delta$,

$$\xi^{\gamma+n-1} = 0.22^{0.5+3} \simeq 0.4993 \neq 1.2346 \simeq \frac{1}{0.81} = \frac{1}{\lambda},$$

and

$$\mathrm{w}\big(v, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3, \mathrm{m}_4\big) = \frac{|\mathrm{m}_1| + |\mathrm{m}_2| + |\mathrm{m}_3| + |\mathrm{m}_4|}{(v^2+35\pi)(e^v + |\mathrm{m}_1| + |\mathrm{m}_2| + |\mathrm{m}_3| + |\mathrm{m}_4|)} + \cosh(2+v^2),$$

for $v \in \overline{\Delta}$ *and* $\mathrm{m}_i \in \mathbb{R}\,(i = 1, 2, 3, 4)$. *Now,* $\forall v \in \overline{\Delta}$ *and* $(\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3)$, $(\acute{\mathrm{m}}_0, \acute{\mathrm{m}}_1, \acute{\mathrm{m}}_2, \acute{\mathrm{m}}_3) \in \mathbb{R}^4$, *we have*

$$\big|\mathrm{w}\big(v, \mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3\big) - \mathrm{w}\big(v, \acute{\mathrm{m}}_0, \acute{\mathrm{m}}_1, \acute{\mathrm{m}}_2, \acute{\mathrm{m}}_3\big)\big| = \Big|\frac{|\mathrm{m}_0| + |\mathrm{m}_1| + |\mathrm{m}_2| + |\mathrm{m}_3|}{(v^2+35\pi)(e^v + |\mathrm{m}_0| + |\mathrm{m}_1| + |\mathrm{m}_2| + |\mathrm{m}_3|)}$$

$$-\frac{|\acute{m}_0|+|\acute{m}_1|+|\acute{m}_2|+|\acute{m}_3|}{(v^2+35\pi)(e^v+|\acute{m}_0|+|\acute{m}_1|+|\acute{m}_2|+|\acute{m}_3|)}\Bigg| \leq \sum_{i=1}^{n-1}\frac{1}{v^2+35\pi}|m_i-\acute{m}_i|.$$

Thus $\varkappa_i(v) = \frac{1}{v^2+35\pi}$ $(i = 0,1,2,3)$. *Hence* $\varkappa_i^* = \sup_{v\in\overline{\Delta}}\varkappa_i(v) = \frac{1}{35\pi}$, *here* $i = 0,1,2,3$, *and* $\mathring{\kappa} := \varkappa_0^*+\varkappa_1^*+\varkappa_2^*+\varkappa_3^* = \frac{4}{35\pi}$,

$$L_0 = \frac{1}{\Gamma_q\left(\frac{15}{4}+1\right)} + \frac{\Gamma_q(0.5+4)\left(1+|0.81|\,0.72^{0.5+\frac{15}{4}}\right)}{|1-0.81\,0.72^{0.5+4-1}|\Gamma_q(4)\Gamma_q\left(0.5+\frac{15}{4}+1\right)} \simeq \begin{cases} 1.5282, & q=0.13,\\ 0.4775, & q=0.50,\\ 0.0594, & q=0.89,\end{cases}$$

Table 1: Numerical results of L_i, $i = 0,1,2,3$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (84) whenever $q = 0.13$.

n	L_0	L_1	L_2	L_3	$\Upsilon < 1$
			$q = 0.13$		
1	3.3512	3.7355	3.9732	3.7387	0.5384
2	2.3436	2.6536	2.7912	2.6165	0.3785
3	1.9893	2.2758	2.3765	2.2219	0.3224
⋮	⋮	⋮	⋮	⋮	⋮
18	1.5303	1.7875	1.8396	1.7110	0.2499
19	1.5297	1.7869	1.8389	1.7103	0.2498
20	1.5293	1.7864	1.8384	1.7098	0.2497
21	1.5289	1.7861	1.8381	1.7095	0.2496
22	1.5287	1.7858	1.8378	1.7092	0.2496
23	1.5286	1.7857	1.8376	1.7091	0.2496
24	1.5284	1.7855	1.8375	1.7089	0.2496
25	1.5284	1.7855	1.8374	1.7088	0.2496
26	1.5283	1.7854	1.8373	1.7088	0.2495
27	1.5283	1.7853	1.8372	1.7087	0.2495
28	1.5282	1.7853	1.8372	1.7087	0.2495
29	1.5282	1.7853	1.8372	1.7087	0.2495
30	1.5282	1.7853	1.8372	1.7086	0.2495
31	1.5282	1.7853	1.8371	1.7086	0.2495
32	1.5282	1.7852	1.8371	1.7086	0.2495

Table 2: Numerical results of L_i, $i = 0,1,2,3$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (84) whenever $q = 0.50$.

n	L_0	L_1	L_2	L_3	$\Upsilon < 1$
			$q = 0.50$		
1	0.8376	1.6205	1.6607	1.2909	0.1968
2	0.6552	1.2066	1.2513	0.9943	0.1494
3	0.5877	1.0668	1.1076	0.8866	0.1327
⋮	⋮	⋮	⋮	⋮	⋮
16	0.4788	0.8866	0.9066	0.7222	0.1089
17	0.4784	0.8861	0.9061	0.7217	0.1089
18	0.4782	0.8858	0.9056	0.7213	0.1088
19	0.4780	0.8855	0.9053	0.7210	0.1088
20	0.4779	0.8854	0.9051	0.7208	0.1087
21	0.4778	0.8852	0.9050	0.7207	0.1087
22	0.4777	0.8851	0.9049	0.7206	0.1087
23	0.4776	0.8851	0.9048	0.7205	0.1087
24	0.4776	0.8850	0.9047	0.7205	0.1087
25	0.4776	0.8850	0.9047	0.7204	0.1087
26	0.4776	0.8850	0.9046	0.7204	0.1087
27	0.4775	0.8849	0.9046	0.7204	0.1087
28	0.4775	0.8849	0.9046	0.7204	0.1087

$$L_i = \frac{1}{\Gamma_q\left(\frac{15}{4}-\zeta_i+1\right)} + \frac{\Gamma_q(0.5+4)\left(1+|0.81|\,0.72^{\frac{15}{4}+0.5}\right)}{|1-0.81\,0.72^{0.5+4-1}|\Gamma_q(4-\zeta_i)\Gamma_q\left(\frac{15}{4}+0.5+1\right)}, \qquad i = 1,2,3.$$

Tables 1, 2 and 3 show the numerical results of L_0, L_1, L_2, L_3 *and* Υ, *for* $q \in \{0.13, 0.5, 0.89\}$. *Also, graphical representation of the variables of* $\mathbb{F}q$-$\mathbb{DE}$ (84) *for three cases of* $q = 0.13$, 0.5, 0.89 *in Figures 1a, 1b, 1c, 1d and 2. So,*

$$L_1 = \frac{1}{\Gamma_q\left(\frac{15}{4} - \frac{11}{5} + 1\right)} + \frac{\Gamma_q(0.5+4)\left(1+|0.81|\,0.72^{\frac{15}{4}+0.5}\right)}{|1-0.81\,0.72^{0.5+4-1}|\Gamma_q\left(4-\frac{11}{5}\right)\Gamma_q\left(\frac{15}{4}+0.5+1\right)} \simeq \begin{cases} 1.7852, & q = 0.13, \\ 0.8849, & q = 0.50, \\ 0.1735, & q = 0.89, \end{cases}$$

$$L_2 = \frac{1}{\Gamma_q\left(\frac{15}{4} - \frac{5}{3} + 1\right)} + \frac{\Gamma_q(0.5+4)\left(1+|0.81|\,0.72^{\frac{15}{4}+0.5}\right)}{|1-0.81\,0.72^{0.5+4-1}|\Gamma_q\left(4-\frac{5}{3}\right)\Gamma_q\left(\frac{15}{4}+0.5+1\right)} \simeq \begin{cases} 1.8371, & q = 0.13, \\ 0.9046, & q = 0.50, \\ 0.1713, & q = 0.89, \end{cases}$$

$$L_3 = \frac{1}{\Gamma_q\left(\frac{15}{4} - \frac{6}{7} + 1\right)} + \frac{\Gamma_q(0.5+4)\left(1+|0.81|\,0.72^{\frac{15}{4}+0.5}\right)}{|1-0.81\,0.72^{0.5+4-1}|\Gamma_q\left(4-\frac{6}{7}\right)\Gamma_q\left(\frac{15}{4}+0.5+1\right)} \simeq \begin{cases} 1.7086, & q = 0.13, \\ 0.7204, & q = 0.50, \\ 0.1185, & q = 0.89, \end{cases}$$

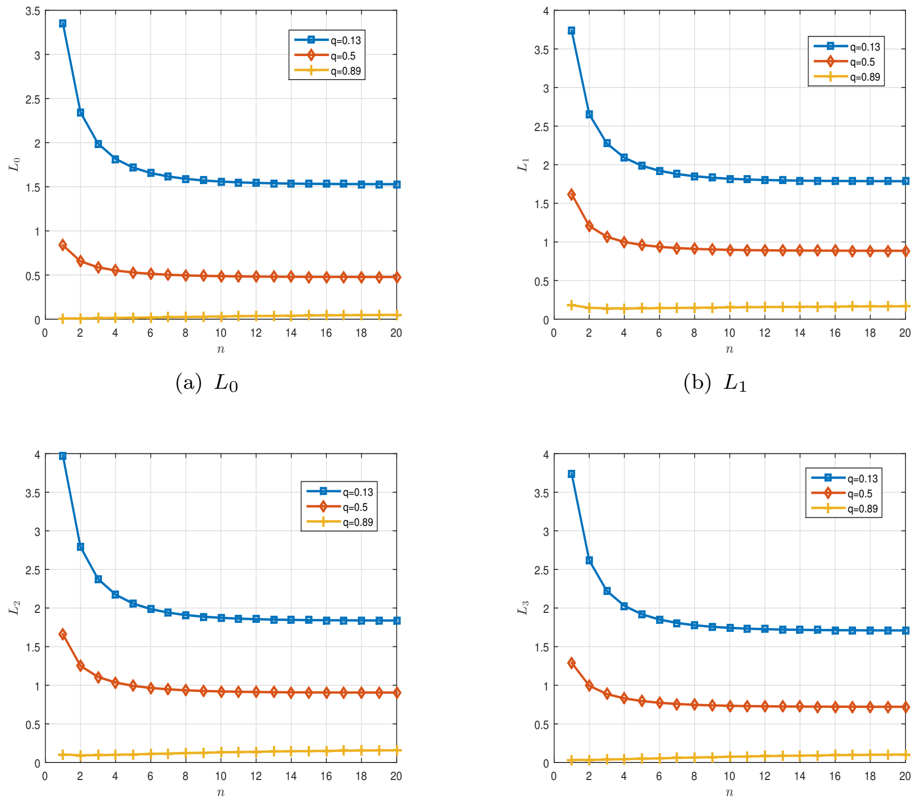
and so from (27), we obtain

Table 3: Numerical results of L_i, $i = 0, 1, 2, 3$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (84) whenever $q = 0.89$.

n	L_0	L_1	L_2	L_3	$\Upsilon < 1$
			$q = 0.89$		
1	0.0078	0.1815	0.1044	0.0328	0.0119
2	0.0090	0.1468	0.0927	0.0334	0.0103
3	0.0111	0.1394	0.0942	0.0376	0.0103
⋮	⋮	⋮	⋮	⋮	⋮
21	0.0499	0.1675	0.1595	0.1047	0.0175
22	0.0509	0.1681	0.1608	0.1062	0.0177
23	0.0518	0.1687	0.1619	0.1075	0.0178
24	0.0526	0.1692	0.1630	0.1087	0.0180
⋮	⋮	⋮	⋮	⋮	⋮
37	0.0579	0.1726	0.1695	0.1163	0.0188
38	0.0581	0.1727	0.1697	0.1166	0.0188
39	0.0582	0.1727	0.1699	0.1168	0.0188
40	0.0583	0.1728	0.1700	0.1170	0.0189
41	0.0585	0.1729	0.1702	0.1171	0.0189
42	0.0586	0.1730	0.1703	0.1173	0.0189
⋮	⋮	⋮	⋮	⋮	⋮
59	0.0593	0.1734	0.1712	0.1184	0.0190
60	0.0593	0.1734	0.1712	0.1184	0.0190
61	0.0593	0.1734	0.1712	0.1184	0.0190
62	0.0593	0.1734	0.1712	0.1184	0.0190
63	0.0594	0.1734	0.1712	0.1184	0.0190
64	0.0594	0.1734	0.1713	0.1184	0.0190
65	0.0594	0.1734	0.1713	0.1184	0.0190
66	0.0594	0.1735	0.1713	0.1185	0.0190
67	0.0594	0.1735	0.1713	0.1185	0.0190
68	0.0594	0.1735	0.1713	0.1185	0.0190
69	0.0594	0.1735	0.1713	0.1185	0.0190

$$\Upsilon = \left(L_0 + \sum_{i=1}^{n-1} L_i\right)\frac{4}{35\pi} \simeq \left\{\begin{array}{ll} 0.2495, & q = 0.13, \\ 0.1087, & q = 0.50, \\ 0.0189, & q = 0.89, \end{array}\right\} < 1.$$

Hence by Theorem 3.1, the problem (84) has a unique solution on Δ.



(a) L_0 (b) L_1

(c) L_2 (d) L_3

Figure 1: Graphical representation of L_i of $\mathbb{F}q$-$\mathbb{DE}$ (84) for $q = 0.13,\ 0.5,\ 0.89$.

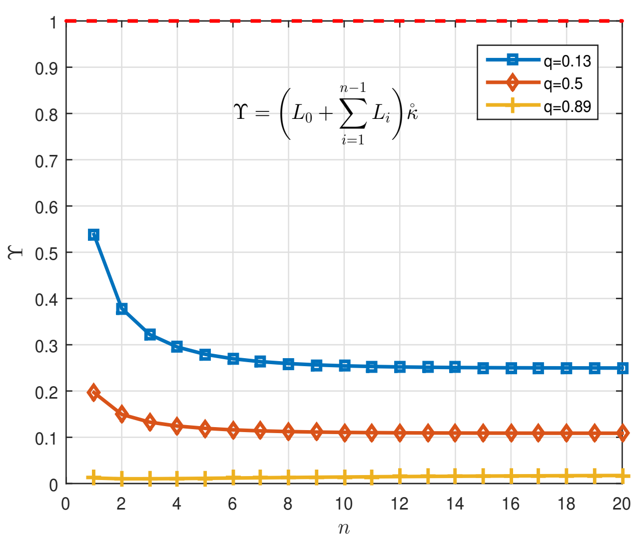


Figure 2: Graphical representation of Υ for $q \in \{0.13,\ 0.5,\ 0.89\}$.

Example 4.2. *Let us consider the following q-fractional boundary value problem* $\mathbb{F}q$-$\mathbb{DE}$,

$$\begin{cases} \mathbb{D}_q^{13/5}\mathfrak{m}(v) = \frac{\tan \mathfrak{m}(v) + \sin\left(\mathbb{D}_q^{7/5}\mathfrak{m}(v) + \mathbb{D}_q^{1/6}\mathfrak{m}(v)\right)}{\left(17\sqrt{\pi}+e^{|v|}\right)^2} & v \in \overline{\Delta} \\ \begin{cases} \mathfrak{m}(0) = 2.5, \\ \mathfrak{m}'(0) = \mathfrak{m}''(0) = 0, \\ \mathbb{I}_q^{1/7}\mathfrak{m}(1) = 0.75\,\mathbb{I}_q^{1/7}\mathfrak{m}\,(0.34)\,, \end{cases} \end{cases} \tag{85}$$

for $q \in \left\{\frac{21}{100}, \frac{1}{2}, \frac{83}{100}\right\}$. *It goes without saying* $\zeta_0 = \frac{13}{5} < n = 3$, $\zeta_1 = \frac{7}{5} < \zeta_0$, $\zeta_2 = \frac{1}{6} < \zeta_1$, $\mathfrak{m}_\circ = 2.5$, $\gamma = \frac{1}{7} \in \Delta$, $\lambda = 0.75 \neq 0$, $\xi = 0.34 \in \Delta$,

$$\xi^{\gamma+n-1} = 0.34^{\frac{1}{7}+3} \simeq 0.0991 \neq 1.3334 \simeq \frac{1}{0.75} = \frac{1}{\lambda},$$

and

$$\mathrm{w}\left(v, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3\right) = \frac{\tan \mathrm{m}_1 + \sin(\mathrm{m}_2 + \mathrm{m}_3)}{\left(17\sqrt{\pi}+e^{|v|}\right)^2},$$

for $v \in \overline{\Delta}$ *and* $\mathrm{m}_i \in \mathbb{R}\,(i = 1, 2, 3)$. *Now,* $\forall v \in \overline{\Delta}$ *and* $(\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2) \in \mathbb{R}^3$, *we have*

$$\left|\mathrm{w}\left(v, \mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2\right)\right| = \left|\frac{\tan \mathrm{m}_1 + \sin(\mathrm{m}_2 + \mathrm{m}_3)}{\left(17\sqrt{\pi}+e^{|v|}\right)^2}\right| \leq \sum_{i=0}^{n-1} \frac{2}{\left(17\sqrt{\pi}+e^{|v|}\right)^2}.$$

Thus $\varrho(v) = \frac{2}{\left(17\sqrt{\pi}+e^{|v|}\right)^2}$. *Hence* $\varrho^* = \sup_{v\in\overline{\Delta}} \varrho(v) = \frac{2}{\left(17\sqrt{\pi}+1\right)^2}$, *and*

$$L_0 = \frac{1}{\Gamma_q\left(\frac{13}{5}+1\right)} + \frac{\Gamma_q\left(\frac{1}{7}+3\right)\left(1+|0.75|\,0.34^{\frac{1}{7}+\frac{13}{5}}\right)}{|1-0.75\,0.34^{\frac{1}{7}+3-1}|\Gamma_q(3)\Gamma_q\left(\frac{1}{7}+\frac{13}{5}+1\right)} \simeq \begin{cases} 1.4426, & q = 0.21, \\ 0.8117, & q = 0.50, \\ 0.2483, & q = 0.83, \end{cases}$$

Table 4: Numerical results of L_i, $i = 0, 1, 2$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (85) whenever $q = 0.21$.

n	L_0	L_1	L_2	$\Upsilon < 1$
		$q = 0.21$		
1	2.7322	2.4806	2.8024	0.0496
2	1.6439	1.5740	1.6884	0.0304
3	1.5007	1.4559	1.5419	0.0279
4	1.4615	1.4238	1.5018	0.0272
5	1.4489	1.4136	1.4890	0.0269
6	1.4448	1.4102	1.4847	0.0269
7	1.4434	1.4091	1.4832	0.0268
8	1.4429	1.4087	1.4827	0.0268
9	1.4427	1.4086	1.4826	0.0268
10	1.4427	1.4085	1.4825	0.0268
11	1.4426	1.4085	1.4825	0.0268
12	1.4426	1.4085	1.4825	0.0268
13	1.4426	1.4085	1.4825	0.0268

$$L_i = \frac{1}{\Gamma_q\left(\frac{13}{5}-\zeta_i+1\right)} + \frac{\Gamma_q\left(\frac{1}{7}+3\right)\left(1+|0.75|\,0.34^{\frac{13}{5}+\frac{1}{7}}\right)}{\left|1-0.75\,0.34^{\frac{1}{7}+3-1}\right|\Gamma_q(3-\zeta_i)\Gamma_q\left(\frac{13}{5}+\frac{1}{7}+1\right)}, \qquad i = 1, 2, 3.$$

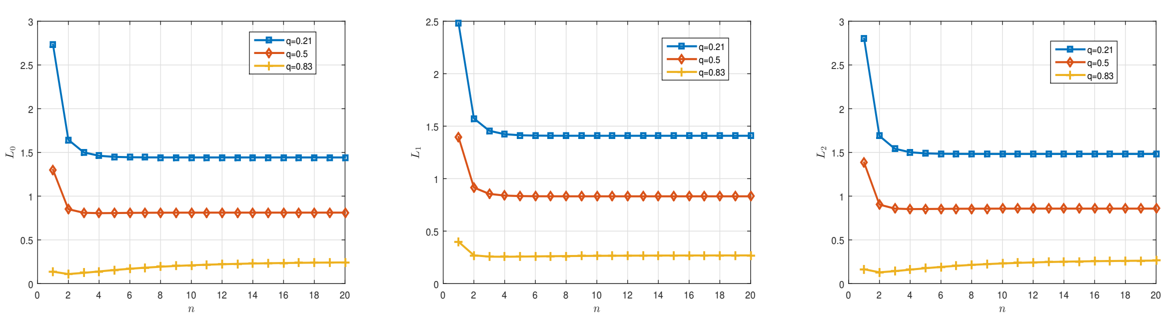
Table 5: Numerical results of L_i, $i = 0, 1, 2$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (85) whenever $q = 0.50$.

n	L_0	L_1	L_2	$\Upsilon < 1$
		$q = 0.50$		
1	1.3004	1.3962	1.3842	0.0253
2	0.8506	0.9155	0.9029	0.0165
3	0.8098	0.8549	0.8578	0.0156
4	0.8053	0.8388	0.8519	0.0155
5	0.8068	0.8339	0.8529	0.0154
6	0.8087	0.8323	0.8546	0.0155
7	0.8100	0.8318	0.8558	0.0155
8	0.8108	0.8317	0.8566	0.0155
9	0.8112	0.8316	0.8570	0.0155
10	0.8114	0.8316	0.8572	0.0155
11	0.8115	0.8316	0.8573	0.0155
12	0.8116	0.8316	0.8574	0.0155
13	0.8116	0.8316	0.8574	0.0155
14	0.8116	0.8316	0.8574	0.0155
15	0.8117	0.8316	0.8574	0.0155
16	0.8117	0.8316	0.8574	0.0155
17	0.8117	0.8316	0.8574	0.0155

Tables 1, 2 and 6 show the numerical results of L_0, L_1, L_2 and Υ, for $q \in \{0.21, 0.5, 0.83\}$. Also, graphical representation of the variables of $\mathbb{F}q\mathbb{DE}$ (85) for three cases of $q = 0.21, 0.5, 0.83$ in Figures 3a, 3b and 3c. So,

$$L_1 = \frac{1}{\Gamma_q\left(\frac{13}{5} - \frac{7}{5} + 1\right)} + \frac{\Gamma_q\left(\frac{1}{7} + 3\right)\left(1 + |0.75|\, 0.34^{\frac{13}{5} + \frac{1}{7}}\right)}{\left|1 - 0.75\, 0.34^{\frac{1}{7} + 3 - 1}\right| \Gamma_q\left(3 - \frac{7}{5}\right)\Gamma_q\left(\frac{13}{5} + \frac{1}{7} + 1\right)} \simeq \begin{cases} 1.4085, & q = 0.21, \\ 0.8316, & q = 0.50, \\ 0.2694, & q = 0.83, \end{cases}$$

$$L_2 = \frac{1}{\Gamma_q\left(\frac{13}{5} - \frac{1}{6} + 1\right)} + \frac{\Gamma_q\left(\frac{1}{7} + 3\right)\left(1 + |0.75|\, 0.34^{\frac{13}{5} + \frac{1}{7}}\right)}{\left|1 - 0.75\, 0.34^{\frac{1}{7} + 3 - 1}\right| \Gamma_q\left(3 - \frac{1}{6}\right)\Gamma_q\left(\frac{13}{5} + \frac{1}{7} + 1\right)} \simeq \begin{cases} 1.4825, & q = 0.21, \\ 0.8574, & q = 0.50, \\ 0.2688, & q = 0.83. \end{cases}$$



(a) L_0 (b) L_1 (c) L_2

Figure 3: Graphical representation of L_i of $\mathbb{F}q$-$\mathbb{DE}$ (85) for $q = 0.21, 0.5, 0.83$.

In the third example we examine Theorem 3.3.

Table 6: Numerical results of L_i, $i=0,1,2$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (85) whenever $q=0.83$.

n	L_0	L_1	L_2	$\Upsilon<1$
		$q=0.83$		
1	0.1361	0.3983	0.1642	0.0043
2	0.1092	0.2695	0.1290	0.0031
3	0.1226	0.2568	0.1423	0.0032
⋮	⋮	⋮	⋮	⋮
25	0.2459	0.2691	0.2664	0.0048
26	0.2463	0.2691	0.2668	0.0048
27	0.2466	0.2692	0.2672	0.0048
28	0.2469	0.2692	0.2675	0.0049
29	0.2471	0.2692	0.2677	0.0049
30	0.2473	0.2692	0.2679	0.0049
31	0.2475	0.2693	0.2680	0.0049
32	0.2476	0.2693	0.2682	0.0049
33	0.2477	0.2693	0.2683	0.0049
⋮	⋮	⋮	⋮	⋮
42	0.2482	0.2693	0.2687	0.0049
43	0.2482	0.2693	0.2687	0.0049
44	0.2482	0.2693	0.2688	0.0049
45	0.2482	0.2694	0.2688	0.0049
46	0.2482	0.2694	0.2688	0.0049
47	0.2483	0.2694	0.2688	0.0049
48	0.2483	0.2694	0.2688	0.0049
49	0.2483	0.2694	0.2688	0.0049

Example 4.3. *Consider the following $\mathbb{F}q$-$\mathbb{DE}$ with the boundary condition,* $q\in\{0.1,\,0.5,\,0.9\}$,

$$\begin{cases} \mathbb{D}_q^{17/5}\mathfrak{m}(v) = \frac{e^{-v^2}}{25\pi+e^{|v|}}\Big(e^{-|\mathfrak{m}(v)|} + \tan \mathbb{D}_q^{8/3}\mathfrak{m}(v) + \sin^2\Big(\mathbb{D}_q^{11/7}\mathfrak{m}(v)\Big) + \tan^{-1}\Big(\mathbb{D}_q^{9/10}\mathfrak{m}(v)\Big)\Big) + \ln(5+v^2), \\ \quad \begin{cases} \mathfrak{m}(0) = 1.5\sqrt{3}, \\ \mathfrak{m}'(0) = \mathfrak{m}''(0) = 0, \\ \mathbb{I}_q^{4/5}\mathfrak{m}(1) = 0.12\,\mathbb{I}_q^{4/5}\mathfrak{m}\,(0.47)\,, \end{cases} \end{cases} \tag{86}$$

for $v\in\overline{\Delta}$. *It goes without saying* $\zeta_0=\frac{17}{5}<n=4$, $\zeta_1=\frac{8}{3}<\zeta_0$, $\zeta_2=\frac{11}{7}<\zeta_1$, $\zeta_3=\frac{9}{10}<\zeta_2$, $\mathfrak{m}_\circ=1.5\sqrt{3}$, $\gamma=\frac{4}{5}\in\Delta$, $\lambda=0.12\neq 0$, $\xi=0.47\in\Delta$,

$$\xi^{\gamma+n-1} = 0.47^{0.8+3} \simeq 0.0567 \neq 8.3333 \simeq \frac{1}{0.12} = \frac{1}{\lambda},$$

and

$$\mathrm{w}\big(v,\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3,\mathrm{m}_4\big) = \frac{e^{-v^2}}{25\pi+e^{|v|}}\Big(e^{-|\mathrm{m}_1|} + \tan\mathrm{m}_2 + \sin^2\mathrm{m}_3 + \tan^{-1}\mathrm{m}_4\Big) + \ln(2+v^2),$$

for $v\in\overline{\Delta}$ *and* $\mathrm{m}_i\in\mathbb{R}\,(i=1,2,3,4)$. *Now,* $\forall v\in\overline{\Delta}$ *and* $(\mathrm{m}_0,\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3)$, $(\acute{\mathrm{m}}_0,\acute{\mathrm{m}}_1,\acute{\mathrm{m}}_2,\acute{\mathrm{m}}_3)\in\mathbb{R}^4$, *we have*

$$\begin{aligned} &\big|\mathrm{w}\big(v,\mathrm{m}_0,\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3\big) - \mathrm{w}\big(v,\acute{\mathrm{m}}_0,\acute{\mathrm{m}}_1,\acute{\mathrm{m}}_2,\acute{\mathrm{m}}_3\big)\big| \\ &\qquad = \Bigg|\frac{e^{-v^2}}{25\pi+e^{|v|}}\Big(e^{-|\mathrm{m}_1|} + \tan\mathrm{m}_2 + \sin^2\mathrm{m}_3 + \tan^{-1}\mathrm{m}_4\Big) - \frac{e^{-v^2}}{25\pi+e^{|v|}}\Big(e^{-|\acute{\mathrm{m}}_1|} \\ &\qquad\quad + \tan\acute{\mathrm{m}}_2 + \sin^2\acute{\mathrm{m}}_3 + \tan^{-1}\acute{\mathrm{m}}_4\Big)\Bigg| \leq \sum_{i=1}^{n-1}\frac{e^{-v^2}}{25\pi+e^{|v|}}|\mathrm{m}_i-\acute{\mathrm{m}}_i|. \end{aligned}$$

Thus $\varkappa_i(v)=\frac{e^{-v^2}}{25\pi+e^{|v|}}\ (i=0,1,2,3)$. *Hence*

$$\varkappa_i^* = \sup_{v\in\overline{\Delta}}\varkappa_i(v) = \frac{1}{25\pi+1}, \qquad i=0,1,2,3,$$

and $\mathring{\kappa} := \varkappa_0^* + \varkappa_1^* + \varkappa_2^* + \varkappa_3^* = \frac{4}{25\pi+1}$,

$$L_0 = \frac{1}{\Gamma_q\left(\frac{17}{5}+1\right)} + \frac{\Gamma_q\left(\frac{4}{5}+4\right)\left(1+|0.12|\, 0.47^{\frac{4}{5}+\frac{17}{5}}\right)}{|1-0.12\, 0.47^{\frac{4}{5}+4-1}|\Gamma_q(4)\Gamma_q\left(\frac{4}{5}+\frac{17}{5}+1\right)} \simeq \begin{cases} 1.5734, & q=0.1, \\ 0.5330, & q=0.5, \\ 0.0699, & q=0.9, \end{cases}$$

Table 7: Numerical results of L_i, $i=0,1,2,3$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (86) whenever $q=0.1$.

n	L_0	L_1	L_2	L_3	$\Upsilon < 1$
			$q = 0.1$		
1	1.6269	1.4023	1.8656	1.7791	0.3356
2	1.5983	1.3791	1.8327	1.7477	0.3298
3	1.5850	1.3697	1.8176	1.7332	0.3272
⋮	⋮	⋮	⋮	⋮	⋮
4	1.5788	1.3655	1.8106	1.7264	0.3259
5	1.5759	1.3636	1.8073	1.7233	0.3254
6	1.5746	1.3627	1.8058	1.7218	0.3251
7	1.5739	1.3623	1.8051	1.7212	0.3250
8	1.5737	1.3621	1.8048	1.7208	0.3249
9	1.5735	1.3620	1.8046	1.7207	0.3249
10	1.5734	1.3619	1.8045	1.7206	0.3249
11	1.5734	1.3619	1.8045	1.7206	0.3249
12	1.5734	1.3619	1.8045	1.7206	0.3249

Table 8: Numerical results of L_i, $i=0,1,2,3$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (86) whenever $q=0.50$.

n	L_0	L_1	L_2	L_3	$\Upsilon < 1$
			$q = 0.50$		
1	0.4470	0.6676	0.8883	0.7068	0.1363
2	0.4887	0.6297	0.9189	0.7543	0.1404
3	0.5104	0.6135	0.9343	0.7786	0.1427
4	0.5215	0.6060	0.9421	0.7909	0.1439
5	0.5272	0.6024	0.9461	0.7972	0.1445
6	0.5301	0.6006	0.9481	0.8003	0.1448
7	0.5315	0.5997	0.9491	0.8020	0.1450
8	0.5322	0.5993	0.9497	0.8028	0.1450
9	0.5326	0.5991	0.9499	0.8032	0.1451
10	0.5328	0.5990	0.9501	0.8034	0.1451
11	0.5329	0.5990	0.9502	0.8035	0.1451
12	0.5329	0.5989	0.9502	0.8035	0.1451
13	0.5330	0.5989	0.9502	0.8036	0.1451
14	0.5330	0.5989	0.9502	0.8036	0.1451
15	0.5330	0.5989	0.9502	0.8036	0.1451

$$L_i = \frac{1}{\Gamma_q\left(\frac{17}{5}-\zeta_i+1\right)} + \frac{\Gamma_q(0.8+4)\left(1+|0.12|\, 0.47^{\frac{17}{5}+0.8}\right)}{|1-0.12\, 0.47^{0.8+4-1}|\Gamma_q(4-\zeta_i)\Gamma_q\left(\frac{17}{5}+0.8+1\right)}, \qquad i=1,2,3.$$

Tables 7, 8 and 9 show the numerical results of L_0, L_1, L_2, L_3 *and* Υ, *for* $q \in \{0.1, 0.5, 0.9\}$. *Also, graphical representation of the variables of* $\mathbb{F}q$-$\mathbb{DE}$ (86) *for three cases of* $q = 0.1, 0.5, 0.9$ *in Figures 4a, 4b, 4c, 4d and 5. So,*

$$L_1 = \frac{1}{\Gamma_q\left(\frac{17}{5}-\frac{8}{3}+1\right)} + \frac{\Gamma_q(0.8+4)\left(1+|0.12|\, 0.47^{\frac{17}{5}+0.8}\right)}{|1-0.12\, 0.47^{0.8+4-1}|\Gamma_q\left(4-\frac{8}{3}\right)\Gamma_q\left(\frac{17}{5}+0.8+1\right)} \simeq \begin{cases} 1.3619, & q=0.1, \\ 0.5989, & q=0.5, \\ 0.1055, & q=0.9, \end{cases}$$

$$L_2 = \frac{1}{\Gamma_q\left(\frac{17}{5}-\frac{11}{7}+1\right)} + \frac{\Gamma_q(0.8+4)\left(1+|0.12|\,0.47^{\frac{17}{5}+0.8}\right)}{|1-0.12\,0.47^{0.8+4-1}|\Gamma_q\left(4-\frac{11}{7}\right)\Gamma_q\left(\frac{17}{5}+0.8+1\right)} \simeq \begin{cases} 1.8045, & q=0.1, \\ 0.9502, & q=0.5, \\ 0.1775, & q=0.9, \end{cases}$$

$$L_3 = \frac{1}{\Gamma_q\left(\frac{17}{5}-\frac{9}{10}+1\right)} + \frac{\Gamma_q(0.8+4)\left(1+|0.12|\,0.47^{\frac{17}{5}+0.8}\right)}{|1-0.12\,0.47^{0.8+4-1}|\Gamma_q\left(4-\frac{9}{10}\right)\Gamma_q\left(\frac{17}{5}+0.8+1\right)} \simeq \begin{cases} 1.7206, & q=0.1, \\ 0.8036, & q=0.5, \\ 0.1376, & q=0.9, \end{cases}$$

and so from (27), we obtain

Table 9: Numerical results of L_i, $i=0,1,2,3$ and Υ of $\mathbb{F}q$-$\mathbb{DE}$ (86) whenever $q=0.9$.

n	L_0	L_1	L_2	L_3	$\Upsilon < 1$
			$q = 0.9$		
1	0.0048	0.1752	0.0658	0.0233	0.0135
2	0.0074	0.1558	0.0777	0.0315	0.0137
3	0.0105	0.1444	0.0884	0.0396	0.0142
⋮	⋮	⋮	⋮	⋮	⋮
20	0.0558	0.1090	0.1637	0.1190	0.0225
21	0.0572	0.1086	0.1651	0.1208	0.0227
22	0.0584	0.1082	0.1664	0.1225	0.0229
⋮	⋮	⋮	⋮	⋮	⋮
34	0.0668	0.1062	0.1746	0.1336	0.0242
35	0.0671	0.1061	0.1749	0.1340	0.0242
36	0.0674	0.1060	0.1752	0.1344	0.0243
37	0.0677	0.1060	0.1755	0.1348	0.0243
38	0.0679	0.1059	0.1757	0.1351	0.0244
⋮	⋮	⋮	⋮	⋮	⋮
46	0.0692	0.1056	0.1769	0.1367	0.0246
47	0.0693	0.1056	0.1770	0.1369	0.0246
48	0.0694	0.1056	0.1771	0.1370	0.0246
49	0.0695	0.1056	0.1772	0.1371	0.0246
50	0.0696	0.1055	0.1772	0.1372	0.0246
51	0.0696	0.1055	0.1773	0.1373	0.0246
52	0.0697	0.1055	0.1773	0.1373	0.0246
53	0.0698	0.1055	0.1774	0.1374	0.0246
54	0.0698	0.1055	0.1774	0.1375	0.0247
55	0.0698	0.1055	0.1775	0.1375	0.0247
56	0.0699	0.1055	0.1775	0.1376	0.0247
57	0.0699	0.1055	0.1775	0.1376	0.0247
58	0.0699	0.1055	0.1776	0.1376	0.0247

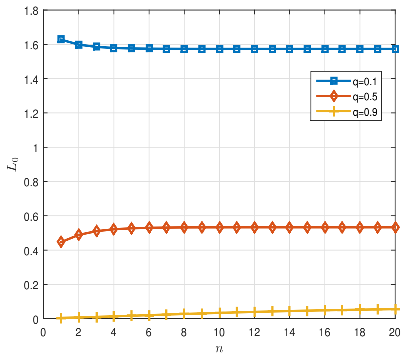
$$\Upsilon = \left(L_0 + \sum_{i=1}^{n-1} L_i\right)\frac{4}{25\pi+1} \simeq \left\{\begin{array}{ll} 0.3249, & q=0.1, \\ 0.1451, & q=0.5, \\ 0.0247, & q=0.9, \end{array}\right\} < 1.$$

Now by using (62), we have

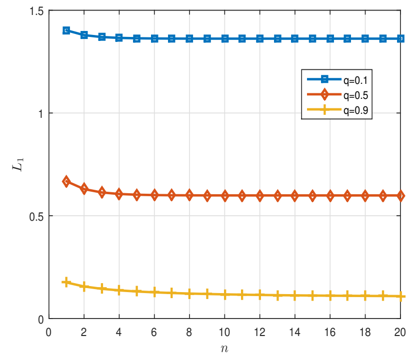
$$\Upsilon_1 = \left(\frac{1}{\Gamma_q\left(\frac{17}{5}+1\right)} + \frac{1}{\Gamma_q\left(\frac{17}{5}-\frac{8}{3}+1\right)} + \frac{1}{\Gamma_q\left(\frac{17}{5}-\frac{11}{7}+1\right)} + \frac{1}{\Gamma_q\left(\frac{17}{5}-\frac{9}{10}+1\right)}\right)\frac{4}{25\pi+1} \simeq \left\{\begin{array}{ll} 0.1703, & q=0.1, \\ 0.0821, & q=0.5, \\ 0.0151, & q=0.9, \end{array}\right\} < 1.$$

On the other hand $\forall v \in \overline{\Delta}$ *and* $(\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3) \in \mathbb{R}^4$, *we have*

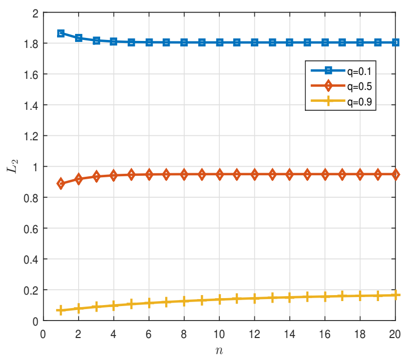
$$\left|\mathrm{w}\left(v, \mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3\right)\right| = \left|\frac{e^{-v^2}}{25\pi+e^{|v|}}\left(e^{-|\mathrm{m}_1|} + \tan \mathrm{m}_2 + \sin^2 \mathrm{m}_3 + \tan^{-1} \mathrm{m}_4\right)\right| \le \sum_{i=1}^{n-1} \frac{2e^{-v^2}}{25\pi+e^{|v|}}.$$



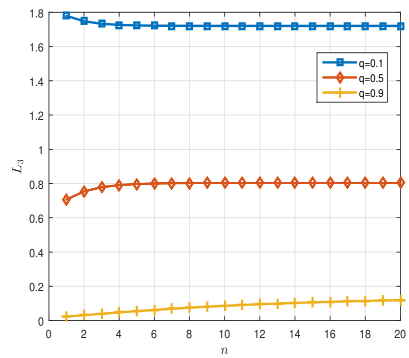
(a) L_0



(b) L_1



(c) L_2



(d) L_3

Figure 4: Graphical representation of L_i of $\mathbb{F}q$-$\mathbb{DE}$ (86) for $q = 0.1,\, 0.5,\, 0.9$.

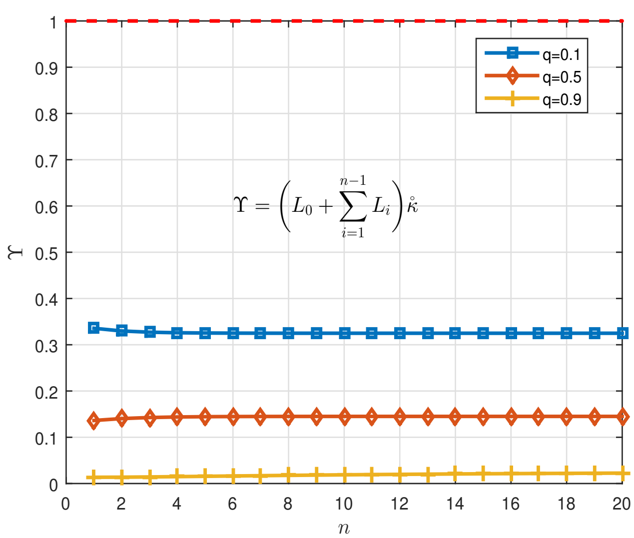


Figure 5: Graphical representation of Υ for $q \in \{0.1,\, 0.5,\, 0.9\}$.

Table 10: Numerical results of Υ_1 and Υ_2 of $\mathbb{F}q$-$\mathbb{DE}$ (86) whenever $q \in \{0.1, 0.5, 0.9\}$.

n	$\Upsilon_1 < 1$	$\Gamma_q(\gamma+1)$	$\Upsilon_2 < \hat{r}$	$\Upsilon_1 < 1$	$\Gamma_q(\gamma+1)$	$\Upsilon_2 < \hat{r}$	$\Upsilon_1 < 1$	$\Gamma_q(\gamma+1)$	$\Upsilon_2 < \hat{r}$
		$q = 0.1$			**$q = 0.5$**			**$q = 0.9$**	
1	0.1702	1.1705	16.1617	0.0745	2.1519	16.0055	0.0086	8.5850	57.3956
2	0.1703	1.1712	16.1426	0.0782	2.1986	14.9747	0.0088	9.1067	33.2256
3	0.1703	1.1712	16.1397	0.0802	2.2206	14.5266	0.0092	9.4924	23.0885
4	0.1703	1.1712	16.1389	0.0811	2.2313	14.3166	0.0096	9.7931	17.8307
5	0.1703	1.1712	16.1386	0.0816	2.2366	14.2149	0.0100	10.0355	14.7225
6	0.1703	1.1712	16.1385	0.0819	2.2392	14.1648	0.0104	10.2353	12.7156
7	0.1703	1.1712	16.1384	0.0820	2.2405	14.1400	0.0107	10.4027	11.3353
8	0.1703	1.1712	16.1384	0.0821	2.2412	14.1276	0.0111	10.5448	10.3401
9	0.1703	1.1712	16.1384	0.0821	2.2415	14.1215	0.0115	10.6664	9.5960
10	0.1703	1.1712	16.1384	0.0821	2.2417	14.1184	0.0118	10.7713	9.0236
11	0.1703	1.1712	16.1384	0.0821	2.2417	14.1168	0.0121	10.8623	8.5730
12	0.1703	1.1712	16.1383	0.0821	2.2418	14.1161	0.0123	10.9417	8.2115
13	0.1703	1.1712	16.1383	0.0821	2.2418	14.1157	0.0126	11.0113	7.9170
14	0.1703	1.1712	16.1383	0.0821	2.2418	14.1155	0.0128	11.0724	7.6739
15	0.1703	1.1712	16.1383	0.0821	2.2418	14.1154	0.0130	11.1263	7.4711
16	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0132	11.1739	7.3002
17	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0134	11.2161	7.1551
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0150	11.5596	6.1612
51	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0150	11.5606	6.1586
52	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5616	6.1564
53	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5624	6.1543
54	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5632	6.1525
55	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5638	6.1508
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
85	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5697	6.1365
86	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5697	6.1364
87	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5697	6.1364
88	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5697	6.1363
89	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5697	6.1363
90	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5698	6.1362
91	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5698	6.1362
92	0.1703	1.1712	16.1383	0.0821	2.2418	14.1153	0.0151	11.5698	6.1362

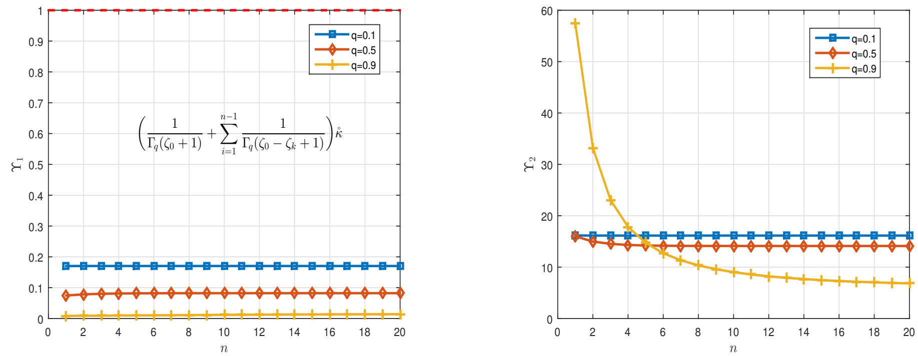
Table 10 shows the numerical results of Υ_1 and Υ_1 for $q \in \{0.1, 0.5, 0.9\}$. One can see the graphical representation of Υ_i of $\mathbb{F}q$-$\mathbb{DE}$ (86) for $q = 0.1,\ 0.5,\ 0.9$ in Figures 6a and 6b. Thus

$$\varrho(v) = \tfrac{2e^{-v^2}}{25\pi + e^{|v|}}, \qquad \varrho^* = \sup_{v \in \overline{\Delta}} \varrho(v) = \tfrac{2}{25\pi+1},$$

and so by employing (63), we obtain

$$\begin{aligned}
\Upsilon_2 = {} & \tfrac{2}{25\pi+1}\left(L_0 + \sum_{i=1}^{n-1} L_i\right) + |1.5\sqrt{3}| + \tfrac{|1.5\sqrt{3}|\,\left|(0.12)\,0.47^{0.8}-1\right|\Gamma_q(0.8+4)\left[\Gamma_q\left(4-\frac{8}{3}\right)+\Gamma_q(4)\right]}{|1-(0.12)\,0.47^{0.8+4-1}|\Gamma_q\left(4-\frac{8}{3}\right)\Gamma_q(4)\Gamma_q(0.8+1)} \\
& + \tfrac{|1.5\sqrt{3}|\,\left|(0.12)\,0.47^{0.8}-1\right|\Gamma_q(0.8+4)\left[\Gamma_q\left(4-\frac{11}{7}\right)+\Gamma_q(4)\right]}{|1-(0.12)\,0.47^{0.8+4-1}|\Gamma_q\left(4-\frac{11}{7}\right)\Gamma_q(4)\Gamma_q(0.8+1)} \\
& + \tfrac{|1.5\sqrt{3}|\,\left|(0.12)\,0.47^{0.8}-1\right|\Gamma_q(0.8+4)\left[\Gamma_q\left(4-\frac{9}{10}\right)+\Gamma_q(4)\right]}{|1-(0.12)\,0.47^{0.8+4-1}|\Gamma_q\left(4-\frac{9}{10}\right)\Gamma_q(4)\Gamma_q(0.8+1)} \simeq \left\{\begin{array}{ll} 16.1383, & q = 0.1, \\ 14.1153, & q = 0.5, \\ 6.1362, & q = 0.9, \end{array}\right\} < \hat{r}.
\end{aligned}$$

By Theorem 3.3, we can state that problem (86) has at least one solution on Δ.



(a) Υ_1 (b) Υ_2

Figure 6: Graphical representation of Υ_i of $\mathbb{F}q$-$\mathbb{DE}$ (86) for $q \in \{0.1, 0.5, 0.9\}$.

5. Conclusion

The $\mathbb{F}q$-$\mathbb{DE}$ has been investigated in this work in detail. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. $\mathbb{F}q$-$\mathbb{DE}$ (10) has been studied on a time scale under some Boundary conditions. An application that describe the motion of a particle in the plane has been provided to support our results' validity and applicability in fields of physics and engineering. The proposed algorithms can help to solve many problems in this regard. It is notable that, by considering the different type of $\mathbb{FDE}$, one can discuss the rest of qualitative properties of such an extended discrete $\mathbb{FDE}$ [49–51] by regarding other generalized models in the next projects.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

MES: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. **HZ**: Actualization, methodology, formal analysis. **MH**: Methodology, validation, formal analysis, actualization, investigation, initial draft and was a major contributor in writing the manuscript. **BR**: Methodology, actualization, formal analysis. All authors read and approved the final manuscript.

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