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Results on fixed points for WC-Mappings satisfying generalized contractive condition in C- metric spaces

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Abstract: In the present paper, we obtained a unique common fixed theorem for WC-mappings (WC-Weakly Compatible) for four self-mappings and also satisfying a generalized type of contractive condition in C- Metric Spaces (Cone -Metric Spaces). Our main aim is in this paper some of the well known results are extends, improves and generalizes , which are existing in the literature.

Key words: C-Metric Space(Cone Metric Space). Common fixed point, WC- mappings(Weakly Compatible-mappings).
AMS subject classifications: 47H10, 54H25.
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1. Introduction

In the non-linear analysis the fixed point theory is one of the important branch. In 2007, Huang and Zhang [4] generalized the concept of a metric space and introduced a new concept, that is C-Metric Space (Cone-Metric Space), and they were replaced the real numbers by an ordered Banach space and also proved some fixed point theorems in C-Metric Space. Subsequently, many authors have been inspired by with these results they extended, improved and generalized in the different ways (see for e.g.,[1-3], [6-11]). Recently, Kumar and Gupta [6] proved a unique common fixed point theorem for two pairs of weakly compatible maps in cone metric spaces. In this paper, we obtained a unique common fixed point theorem for WC-mappings (WC- Weakly Compatible) for four self-mappings in C- Metric Spaces. Our aim is generalized the contractive condition and improve the results.

2. Preliminaries

The following are useful in our main results these are in [2, 4].

Definition 2.1. Let F be a real Banach space and Q be a subset of F. The set Q is called a cone iff
(i). Q is closed, non-empty and $Q \neq \{0\}$;
(ii). Let $\alpha, \beta \in R,\ \alpha, \beta \geq 0,\ u, v \in Q \implies \alpha u + \beta v \in Q$;
(iii). $Q \cap (-Q) = \{0\}$.

For a given cone $Q \subseteq F$, we define a partial ordering " $\leq$ " with respect to Q by $\alpha \leq \beta$ if and only if

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$\beta - \alpha \in Q$. A cone Q is called normal if there exists $L > 0$ such that for all $\alpha, \beta \in Q$,

$$0 \leq \alpha \leq \beta \implies \| \alpha \| \leq L \| \beta \| .....(A)$$

The least positive number L satisfying the above inequality is called the normal constant of Q. While $\alpha << \beta$ stands for β–$\alpha \in intQ$ (interior of Q).

Definition 2.2. Let X be a non empty set of F .Suppose that the map $\rho : X \times X \longrightarrow F$ satisfies :

(1) $0 \leq \rho(\alpha, \beta)$ for all $\alpha, \beta \in X$ and $\rho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;

(2) $\rho(\alpha, \beta) = \rho(\beta, \alpha)$ for all $\alpha, \beta \in X$;

(3) $\rho(\alpha, \beta) \leq \rho(\alpha, \gamma) + \rho(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in X$.

Then ρ is called a cone metric on X and (X, ρ) is called a C-Metric Space(Cone Metric Space).

Definition 2.3. Let (X, ρ) be a C-metric space.We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in F with $0 \ll c$, there is N such that for all $n, m > N$, $\rho(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any $0 \ll c$, there is an N such that for all $n > N$, $\rho(x_n, x) \ll c$, for some fixed x in X. We denote this $x_n \longrightarrow x$ $(n \longrightarrow \infty)$.

Definition 2.4. A C-Metric Space(Cone Metric Space) X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.5. Le A and B be two self maps of a set X. A point x in X are said to be WC- mappings (Weakly Compatible) if they are commute at their coincidence point of A and B, that is $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Definition 2.6. A and B be two self-mappings of a set X. If $z = Ax = Bx$, for some $x \in X$, then x is called a coincidence point of A and B , where z is called coincidence point of coincidence of A and B.

3. Main Results

In this section, we prove a unique common fixed point theorem for WC-mappings for four self-mappings in C-Metric Spaces.
Now we prove our main theorem.

Theorem 3.1. *Let (X, ρ) be a C-Metric Space(Cone Metric Space) and Q be a normal cone. Let A, B, M and N be self-mappings such that:*

$$\rho(Mx, Ny) \leq \lambda_1 \rho(Ax, By) + \lambda_2 [\rho(Ax, Mx) + \rho(By, Ny)]/2 + \lambda_3 [\rho(Ax, Ny) + \rho(By, Mx)]/2. \tag{1}$$

For all $x, y \in X$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 < 1$.

$$N(X) \subseteq A(X), M(X) \subseteq B(X) \text{ and one of } A(X) \text{ or } B(X) \text{ is a complete subspace of } X. \tag{2}$$

$$(A, M) \text{ and } (B, N) \text{ are weakly compatible.} \tag{3}$$

Then A, B, M and N have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Define $y_{2n} = Mx_{2n} = Bx_{2n+1}, y_{2n+1} = Nx_{2n+1} = Ax_{2n+2}$.
$Putting x = x_{2n}, y = x_{2n+1}$, for all n = 1,2,.... By (1) we get that

$$\begin{aligned}
\rho(y_{2n}, y_{2n+1} &= \rho(Mx_{2n}, Nx_{2n+1}) \\
&\leq \lambda_1\rho(Ax_{2n}, Bx_{2n+1}) + \lambda_2[\rho(Ax_{2n}, Mx_{2n}) + \\
&\quad \rho(Bx_{2n+1}, Nx_{2n+1})]/2 + \lambda_3[\rho(Ax_{2n}, Nx_{2n+1}) + \rho(Bx_{2n+1}, Mx_{2n})]/2, \\
&\leq \lambda_1\rho(y_{2n-1}, y_{2n}) + \lambda_2[\rho(y_{2n-1}, y_{2n}) + \rho(y_{2n}, y_{2n+1})]/2 + \lambda_3[\rho(y_{2n-1}, y_{2n+1}) + \rho(y_{2n}, y_{2n+1})]/2, \\
&\leq [\lambda_1 + \lambda_2/2 + \lambda_3/2]\rho(y_{2n-1}, y_{2n}) + [\lambda_2/2 + \lambda_3/2]\rho(y_{2n}, y_{2n+1}), \\
&\leq \frac{[2\lambda_1 + \lambda_2 + \lambda_3]/2}{[2 - (\lambda_2 + \lambda_3)]/2}\rho(y_{2n-1}, y_{2n}), \\
&\leq \frac{[2\lambda_1 + \lambda_2 + \lambda_3]}{[2 - (\lambda_2 + \lambda_3)]}\rho(y_{2n-1}, y_{2n}), \\
\rho(y_{2n}, y_{2n+1} &\leq \alpha\rho(y_{2n-1}, y_{2n}),
\end{aligned}$$

where $\alpha = \frac{[2\lambda_1+\lambda_2+\lambda_3]}{[2-(\lambda_2+\lambda_3)]} < 1$.

Similarly, it can be shown that

$$\rho(y_{2n+1}, y_{2n+2)} \leq \alpha\rho(y_{2n}, y_{2n+1}).$$

Therefore, for all n

$$\rho(y_{n+1}, y_{n+2}) \leq \rho(y_n, y_{n+1}) \leq ... \leq \alpha^{n-1}\rho(y_0, y_1).$$

Now for any $m,\ n\ \to\ \infty$

$$\begin{aligned}
\rho(y_n, y_m) &\leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \leq ... \leq \rho(y_{m-1}, y_m). \\
&\leq [\alpha^n + \alpha^{n-1} + ... + \alpha^{m-1}]\rho(y_0, y_1). \\
&\leq [\alpha^n/1 - \alpha^n]\rho(y_0, y_1).
\end{aligned}$$

From (A) we get that

$$\| \rho(y_n, y_m) \| \leq [\alpha^n/1 - \alpha^n]L \| \rho(y_0, y_1) \| .$$

Which implies that $\rho(y_n, y_m) \to 0 as m, n \to \infty$.
Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that

$$lim_{n\to\infty} y_{2n} = z.$$

$$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} Bx_{2n+1} = z \text{ and } \lim_{n\to\infty} Nx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z.$$

That is $lim_{n\to\infty} Mx_n = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Nx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z$.
Since $N(X) \subseteq A(X)$, there exists a point $u \in X$ such that $Au = z$. Then by (1) we get that

$$\begin{aligned}
\rho(Mu, z) \leq &\rho(Mu, Nx_{2n-1}) + \rho(Nx_{2n-1}, z), \\
&\leq \lambda_1\rho(Au, Bx_{2n-1}) + \lambda_2[\rho(Au, Mu) + \rho(Bx_{2n-1}, Nx_{2n-1})]/2 \\
&\quad + \lambda_3[\rho(Au, Nx_{2n-1}) + \rho(Bx_{2n-1}, Mu)]/2 + \rho(Nx_{2n-1}, z)
\end{aligned}$$

From (A) we get that
$\| \rho(Mu, z) \| \leq \lambda_1 K \| \rho(Au, Bx_{2n-1}) \| + \lambda_2 K \| [\rho(Au, Mu) + \rho(Bx_{2n-1}, Nx_{2n-1})]/2 \| +$
$\lambda_3 K \| [\rho(Au, Nx_{2n-1}) + \rho(Bx_{2n-1}, Mu)]/2 \| + \| \rho(Nx_{2n-1}, z) \|$.
Letting $n \to \infty$ we get that

$$\begin{aligned}\rho(Mu, z) &\leq \lambda_1 \rho(z, z) + \lambda_2 [\rho(z, z) + \rho(zz)]/2 + \lambda_3 [\rho(z, z) + \rho(z, Mu)]/2 + \rho(z, z),\\ &\leq [\lambda_2/2 + \lambda_3/2]\rho(Mu, z),\\ &\leq [\lambda_2 + \lambda_3]/2\rho(Mu, z).\end{aligned}$$

which is a contradiction, because $\lambda_1 + \lambda_2 + \lambda_3 < 1$.
Therefore,

$$Mu = Au = z. \tag{4}$$

Since $M(X) \subseteq B(X)$, there exists a point $v \in X$ such that $Bv = z$. Then by (1) we get that

$$\begin{aligned}\rho(z, Nv) &= \rho(Mu, Nv)\\ &\leq \lambda_1 \rho(Au, Bv) + \lambda_2 [\rho(Au, Mu) + \rho(Bv, Nv)]/2 + \lambda_3 [\rho(Au, Nv) + \rho(Bv, Mu)]/2,\\ &\leq \lambda_1 \rho(z, z) + \lambda_2 [\rho(z, z) + \rho(z, Nv)]/2 + \lambda_3 [\rho(z, Nv) + \rho(z, z)]/2,\\ &\leq [\lambda_2/2 + \lambda_3/2]\rho(z, Nv),\\ \rho(z, Nv) &\leq [\lambda_2 + \lambda_3]/2\rho(z, Nv),\end{aligned}$$

which is a contradiction, because $\lambda_1 + \lambda_2 + \lambda_3 < 1$.
Therefore,

$$Nv = Bv = z. \tag{5}$$

Since, A and M are WC-mappings(Weakly Compatible), then $MAu = AMu$ that is , $Mz = Az$.
Now we shall prove that, z is a fixed point of M. If $Mz \neq z$, then by (1) we get that

$$\begin{aligned}\rho(Mz, z) &= \rho(Mz, Nv)\\ &\leq \lambda_1 \rho(Az, Bv) + \lambda_2 [\rho(Az, Mz) + \rho(Bv, Nv)]/2 + \lambda_3 [\rho(Az, Nv) + \rho(Bv, Mz)]/2,\\ &\leq \lambda_1 \rho(Mz, z) + \lambda_2 [\rho(Mz, Mz) + \rho(z, z)]/2 + \lambda_3 [\rho(Mz, z) + \rho(z, Mz)]/2,\\ &\leq [\lambda_1 + \lambda_3]\rho(Mz, z),\end{aligned}$$

which is a contradiction, because $\lambda_1 + \lambda_2 + \lambda_3 < 1$.
Therefore,

$$Mz = z. \tag{6}$$

Similarly, B and N are WC-mappings (Weakly Compatible), we have, $Nz = Bz$.
Now we shall prove that, z is a fixed point of N. If $Nz \neq z$, then by (1) we get that

$$\begin{aligned}\rho(z, Nz) &= \rho(Mz, Nz)\\ &\leq \lambda_1 \rho(Az, Bz) + \lambda_2 [\rho(Az, Mz) + \rho(Bz, Nz)]/2 + \lambda_3 [\rho(Az, Nz) + \rho(Bz, Mz)]/2,\\ &\leq \lambda_1 \rho(z, Nz) + \lambda_2 [\rho(z, z) + \rho(Nz, Nz)]/2 + \lambda_3 [\rho(z, Nz) + \rho(Nz, z)]/2,\\ &\leq [\lambda_1 + \lambda_3]\rho(z, Nz),\end{aligned}$$

which is a contradiction , because $\lambda_1 + \lambda_2 + \lambda_3 < 1$.
Therefore,

$$Nz = z. \tag{7}$$

Thus, $Mz = Nz = Az = Bz = z$, that is z is a common fixed point of $A, B, MandN$.

Uniqueness: Suppose z_1 is a common fixed point of $A, B, MandN$. Then by (1), we get that

$$\begin{aligned}\rho(z, z_1) &= \rho(Mz, Nz_1)\\ &\le \lambda_1 \rho(Az, Bz_1) + \lambda_2[\rho(Az, Mz) + \rho(Bz_1, Nz_1)]/2 + \lambda_3[\rho(Az, Nz_1) + \rho(Bz_1, Mz)]/2,\\ &\le \lambda_1 \rho(z, z_1) + \lambda_2[\rho(z, z) + \rho(z_1, z_1)]/2 + \lambda_3[\rho(z, z_1) + \rho(z_1, z)]/2,\\ &\le [\lambda_1 + \lambda_3]\rho(z, z_1),\end{aligned}$$

which is a contradiction because $\lambda_1 + \lambda_2 + \lambda_3 < 1$.
Therefore, $z = z_1$.

Hence, z is a unique common fixed point of $A, B, MandN$.
This completes proof of the theorem. □

4. Conclusion

In this research article we generalized the contractive condition and proved generalized results, so that our results are more general improved than the results of [6].

Conflict of interest: The author has declared there is no conflict of interest.

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