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On $\mathcal{Q}_\beta$-pseudo starlike functions

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Abstract: This study introduced a new concept of a class of $\mathcal{Q}_\beta$-pseudo starlike functions with change of notation of the class introduced in [4], using the Quantum approach, involving q-derivative and its integral with investigation of the properties: inclusion, conditions for univalency, coefficient and Fekete-Szego inequalities. Our results are accompanied by their corollaries and implications, which also extend earlier results.

Key words: Analytic and Univalent functions, q-diffrential operator, q-integral operator, β-pseudo starlike functions.

1. Introduction

Fractional calculus, has its origin traced to Liouville's work in 1832, as an established field of mathematical analysis. It is an area of several special functions that have been extensively studied. Recently, fractional calculus remains an area of research, as evidenced by recent investigations in [1–3, 10, 11].

This field of research, which includes both fractional and ordinary derivative operators, remains a research field in complex analysis, specifically within the concept of geometric function theory.

These interesting results have evolved from investigations into quantum (or q-) calculus, which provide alternative concepts on differential and integral operators, and have application in different areas of mathematical physics.

The utilization of q-calculus by researchers in geometry function theory has led to the introduction and investigating series of subclasses of analytic functions. The concept of q-calculus can be traced to Jackson's introduction of q-derivatives and q-integrals in [7, 8] in which applications of q-calculus has been extended across field of humanity, such as control theory, algebraic geometry and many more.

The essential definitions are as follows:
Definition 1.1. The q-*differentiation* of function $f(0) = 0$ and $f'(0) = 1$, for $q \in (0, 1)$ of the form

$$f(\xi) = \xi + \sum_{k=2}^{\infty} d_k \xi^k, \xi \in \mathbb{U} = \{\xi : |\xi| < 1\}, \tag{1}$$

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is defined by

$$\mathbf{D}_q f(0) = f'(0), \mathbf{D}_q f(\xi) = \frac{f(\xi) - f(q\xi)}{\xi(1-q)} (\xi \neq 0), \mathbf{D}_q^2 f(\xi) = \mathbf{D}_q(\mathbf{D}_q f(\xi)), \tag{2}$$

where

$$\mathbf{D}_q f(\xi) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \xi^{k-1}, z\mathbf{D}_q^2 f(\xi) = \sum_{k=2}^{\infty} [k-1]_q [k]_q a_k \xi^{k-1} \tag{3}$$

such that $[k]_q = \frac{1-q^k}{1-q}$ and $q \to 1$, $[k]_q = k$.
Equivalently, the q-integral of the function of the form (3) is gives to be

$$\mathbf{I}_q = \int_0^{\xi} f(w) d_q w = \xi(1-q) \sum_{k=0}^{\infty} q^k f(\xi q^k),$$

provided the q-series converges, see[7, 8].

The class of the function of the form (1) has been defined with many operators by different techniques as in [14], [15] with the investigation of several geometric properties.

The class of β-pseudo starlike functions with change of notation was introduced in [4] with some of properties investigated which motivated this work, where the class of functions is extended to q-calculus by introducing a new class of $\mathcal{Q}_\beta$ pseudo starlike functions define by q-differential operator.

Definition 1.1. For $0 \leq \eta < 1$, $\beta \geq 0$. The class of functions $f(\xi) \in \mathbf{A}$ is said to be in the class of $\mathcal{Q}_\beta$-pseudo starlike functions denoted as $\mathbf{S}_q(\eta, \alpha)$ if and only if

$$\frac{\xi(\mathbf{D}_q f(\xi))^\beta}{f(\xi)} > \eta \tag{4}$$

Remark 1.1. *Varying the parameter q and β, existing subclasses are deduced see [13].*

2. Preliminary Lemmas

Lemma 2.1. *[4]*

Let $p(\xi)$ be holomorphic in Ω with $p(0) = 1$. Suppose that

$$Re\left(1 + \frac{\xi \mathbf{D}_q^1 p(\xi)}{p(\xi)}\right) > \frac{3\eta - 1}{2\eta}.$$

Then

$$Rep(\xi) > 2^{1-\frac{1}{\eta}}, \frac{1}{2} \leq \eta < 1, \xi \in \mathbb{U}. \tag{5}$$

and the constant $2^{1-\frac{1}{\eta}}$ is the best possible.

Lemma 2.2. *[3]*

Let $p \in P$ *, then*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| = \begin{cases} 2(1-\sigma), if \sigma \leq 0 \\ 2, if 0 \leq \sigma \leq 2 \\ 2(\sigma - 1), if \sigma \leq 2 \end{cases}$$

Lemma 2.3. *[3]*

Let $p \in \mathcal{P}$ *, then for any real or complex number* $\mathcal{V}$ *, we have sharp inequalities*

$$\left| p_2 - \mathcal{V} \frac{p_1^2}{2} \right| \leq 2 \max\{1, |1 - \mathcal{V}|\}. \tag{6}$$

Lemma 2.4. *[4]*

Let $p \in P$ *be the class of the function of the form*

$$p(\xi) = 1 + (1-\eta) \sum_{k=1}^{\infty} p_1 \xi^k, \xi \in \mathbb{U}, \tag{7}$$

where $p(0) = 1$ *and* $\Re e p(\eta) > 0$ *,* $\eta \in [0, 1)$ *. Then* $|p_k| \leq 2$ *.*

Lemma 2.5. *[4]*

Let $p_\eta \in P_\eta$ *, if*

$$h(\xi) = [p_\eta(\xi)]^\tau, \tau \in [0, 1].$$

Then $\Phi(0) = 1$ *and* $\Re e[\Phi(\xi)] > \eta^\tau$ *.*

3. Main Results

Theorem 3.1. $\mathbf{S}_q(\eta, \beta) \subset \mathbf{B}_1^q(1 - \frac{1}{\beta}, \eta^{\frac{1}{\beta}})$

Proof. Since $f(\xi) \in \mathbf{S}_q(\eta, \beta)$, there exist $p \in P_\eta(\xi)$ such that

$$\frac{\xi (\mathbf{D}_q f(\xi))^\beta}{f(\xi)} = p(\xi),$$

so that

$$\frac{\xi (\mathbf{D}_q f(\xi))^\beta}{f(\xi)} = \left(\frac{\xi^\beta \mathbf{D}_q f(\xi)}{f(\xi)^{\frac{1}{\beta}}} \right)^\beta = p(\xi),$$

which is equivalent to

$$\frac{\xi^\beta \mathbf{D}_q f(\xi)}{f(\xi)^{\frac{1}{\beta}}} = p(\xi)^{\frac{1}{\beta}}.$$

From Lemma 2.5, we have

$$\Re e \frac{\xi^\beta \mathbf{D}_q f(\xi)}{f(\xi)^{\frac{1}{\beta}}} > \eta^{\frac{1}{\beta}}.$$

Let $\lambda = 1 - \frac{1}{\beta}$, we have $f(\xi) \in \mathbf{B}_1(1 - \frac{1}{\beta}, \eta^{\frac{1}{\beta}})$. □

Corollary 3.1. *For $q \to 1$, The class of the functions $\mathbf{S}_q(\eta,\beta)$ results to β-pseudo starlike functions, with change of notation as studied in [**?**].*

Corollary 3.2. *The class of the functions $\mathbf{S}_q(\eta,\beta)$ are univalent and belong to $\mathbf{B}_1^q(1-\frac{1}{\beta},\eta^{\frac{1}{\beta}})$.*

Theorem 3.2. *The functions $f(\xi)\in\mathbf{S}_q(\eta,\beta)$ has the integral representation*

$$f(\xi)=\mathbf{I}_q\left\{f(w)^{1-\lambda}w^{\lambda-1}p(w)^{1-\lambda}dw.\right\} \tag{8}$$

Proof. Let $f(\xi)\in\mathbf{S}_q(\eta,\beta)$, then there exists $p\in P_\eta(\xi)$ such that

$$\frac{\xi^\beta\mathbf{D}_qf(\xi)}{f(\xi)^{\frac{1}{\beta}}}=p(\xi)^{\frac{1}{\beta}}.$$

Let $\lambda=1-\frac{1}{\beta}$, then

$$\frac{f(\xi)^{\lambda-1}\mathbf{D}_qf(\xi)}{\xi^{\lambda-1}}=p(\xi)^{1-\lambda}.$$

We have

$$\mathbf{D}_qf(\xi)=f(\xi)^{1-\lambda}\xi^{\lambda-1}p(\xi)^{1-\lambda}.$$

Therefore, the condition (8) is obtained.

□

Corollary 3.3. *For $q\to 1$, The class of the functions $\mathbf{S}_q(\eta,\beta)$ results to β-pseudo starlike functions, with change of notation as studied in [4].*

Corollary 3.4. *[6] For $q\to1$, $\beta=1$, the integral representation of starlike functions is obtained .*

Theorem 3.3. *Let $f(\xi)\in\mathbf{S}_q(\eta,\beta)$ satisfies the inequality*

$$\Re e\left\{\beta\frac{\xi\mathbf{D}_q^2(\xi)}{\mathbf{D}_q(\xi)}-\frac{\xi\mathbf{D}_q(\xi)}{f(\xi)}\right\}>-\frac{1+\eta}{2\eta},\xi\in\mathbb{U}.$$

Then $f(\xi)\in\mathbf{S}_q(2^{1-\frac{1}{\eta}},\beta)$, $\frac{1}{2}\le\eta<1$, with the best possible constant $2^{1-\frac{1}{\eta}}$.

Proof. Since $f(\xi)\in\mathbf{S}_q(\eta,\beta)$, such that

$$\frac{\xi(\mathbf{D}_qf(\xi))^\beta}{f(\xi)}=p(\xi).$$

Then

$$\frac{\xi\mathbf{D}_qp(\xi)}{p(\xi)}=1-\frac{\xi\mathbf{D}_qf(\xi)}{f(\xi)}+\beta\frac{\xi\mathbf{D}_q^2f(\xi)}{\mathbf{D}_qf(\xi)},$$

so that

$$\Re e\left[1+\frac{\xi\mathbf{D}_qp(\xi)}{p(\xi)}\right]=\Re e\left(2-\frac{\xi\mathbf{D}_qf(\xi)}{f(\xi)}+\beta\frac{\xi\mathbf{D}_q^2f(\xi)}{\mathbf{D}_qf(\xi)}\right)>\frac{3\eta-1}{2\eta}.$$

We have

$$\beta\frac{\xi\mathbf{D}_q^2 f(\xi)}{\mathbf{D}_q f(\xi)}-\frac{\xi\mathbf{D}_q f(\xi)}{f(\xi)}>-\frac{1+\eta}{2\eta},$$

by Lemma 2.1, we have

$$\Re e\frac{\xi(\mathbf{D}_q f(\xi))^\beta}{f(\xi)}>2^{1-\frac{1}{\eta}},\frac{1}{2}\leq\eta<1$$

□

Corollary 3.5. *For $\eta=\frac{1}{2}$, and $f(\xi)$ satisfies the condition*

$$\Re e\left\{\beta\frac{\xi\mathbf{D}_q^2 f(\xi)}{\mathbf{D}_q f(\xi)}-\frac{\xi\mathbf{D}_q f(\xi)}{f(\xi)}\right\}>-\frac{3}{2},\xi\in\mathbb{U}.$$

Then

$$\Re e\frac{\xi\mathbf{D}_q f(\xi)}{f(\xi)}>\frac{1}{2}.$$

Corollary 3.6. *For $q\to1$, $\eta=\frac{1}{2}$, and $f(\xi)$ satisfies the condition*

$$\Re e\left\{\beta\frac{\xi f''(\xi)}{f'(\xi)}-\frac{\xi f'(\xi)}{f(\xi)}\right\}>-\frac{3}{2},\xi\in\mathbb{U}.$$

Then

$$\Re e\frac{\xi f'(\xi)}{f(\xi)}>\frac{1}{2}.$$

obtained in [4].

Theorem 3.4. *For $f(\xi)\in\mathbf{S}_q(\eta,\beta)$. Then*

$$d_2\leq\frac{2(1-\eta)}{(\beta[2]_q-1)},\tag{9}$$

$$d_3\leq\frac{2(1-\eta)}{([3]_q\beta-1)}\left|\frac{2(1-\eta)(2\beta^2-4\beta+1)}{(2\beta-1)^2}-1\right|\tag{10}$$

,

$$d_4\leq\frac{2(1-\eta)}{([4]_q\beta-1)}+\frac{4(1-\eta)^2}{([4]_q\beta-1)}\left[\frac{6\beta^2-11\beta+2}{(2\beta-1)(3\beta-1)}\right]+\frac{8(1-\eta)^3}{([4]_q\beta-1)}\left[\frac{624\beta^4-80\beta^3+84\beta^2-24\beta+3}{3(2\beta-1)^3(3\beta-1)}\right]\tag{11}$$

$$|d_5|\leq2\Gamma\left|2\frac{\Upsilon}{\Gamma}-1\right|+4|\Pi|\left|2\frac{\Psi}{\Pi}-1\right|+16|\Sigma|.\tag{12}$$

Where

$$\Gamma=\frac{(1-\eta)}{([5]_q\beta-1)},$$

$$\Upsilon = \frac{(1-\eta)}{([5]_q\beta - 1)} \left[\frac{2(24\beta^2 - 7\beta + 1)}{(2\beta - 1)(4\beta - 1)} \right],$$

$$\Pi = \frac{(1-\eta)^2}{([5]_q\beta - 1)} \left[\frac{9\beta^2 - 15\beta + 2)}{2(3\beta - 1)^2} \right],$$

$$\Psi = \frac{(1-\eta)^3}{([5]_q\beta - 1)} \left[\frac{9\beta(\beta-1)(2\beta^2 - 4\beta + 1)}{(2\beta-1)^2(3\beta-1)^2} + \frac{(8\beta^2 - 10\beta + 1)(6\beta^2 - 11\beta + 2)}{(2\beta-1)^2(3\beta-1)(4\beta-1)} \right. \\ \left. - \frac{(6\beta^3 - 16\beta^2 + 8\beta + 1)(6\beta^2 - 11\beta + 2)}{(2\beta-1)^2(3\beta-1)(4\beta-1)} \right]$$

$$\Sigma = \frac{(1-\eta)^3}{([5]_q\beta - 1)} \left[\frac{9\beta(\beta-1)(2\beta^2 - 4\beta + 1)}{(2\beta-1)^2(3\beta-1)^2} + \frac{(8\beta^2 - 10\beta + 1)(6\beta^2 - 11\beta + 2)}{(2\beta-1)^2(3\beta-1)(4\beta-1)} \right. \\ \left. - \frac{(6\beta^3 - 16\beta^2 + 8\beta + 1)(6\beta^2 - 11\beta + 2)}{(2\beta-1)^2(3\beta-1)(4\beta-1)} \right].$$

Proof. Since $f(\xi) \in \mathbf{S}_q(\eta, \beta)$. such that

$$\frac{\xi(\mathbf{D}_q f(\xi))^\beta}{f(\xi)} = \eta + (1-\eta)p(\xi). \tag{13}$$

Then

$$\xi(\mathbf{D}_q f(\xi))^\beta = f(\xi)[\eta + (1-\eta)p(\xi)]. \tag{14}$$

Equation (14) is equivalent to

$$\xi + d_2\beta[2]_q\xi^2 + \left\{ d_3\beta[3]_q + 2\beta(\beta-1)d_2^2[2]_q^2 \right\} \xi^3 \\ + \left\{ d_4\beta[4]_q + 6\beta(\beta-1)d_2d_3[2]_q[3]_q + \beta(\beta-1)(\beta-2)d_2^3[2]_q^3 \frac{[4]_q}{[3]_q} \right\} \xi^4 \\ + \left\{ \beta d_5[5]_q + \frac{[1]_q}{[2]_q}\beta(\beta-1)[16d_2d_3[2]_q[3]_q + 9d_3^2[3]_q] + 6\beta(\beta-1)(\beta-2)d_2^2d_3[2]_q^2[3]_q \right. \\ \left. + \frac{[2]_q}{[3]_q}\beta(\beta-1)(\beta-2)(\beta-3)d_2^4[2]_q^4 \right\} \xi^5 + \cdots$$

$$= \xi + [(1-\eta)p_1 + d_2]\xi^2 + [(1-\eta)p_2 + (1-\eta)p_1d_2 + d_3]\xi^3 + [(1-\eta)p_3 + (1-\eta)p_2d_2 + (1-\eta)p_1d_3 + d_4]\xi^4 \\ + [(1-\eta)p_4 + (1-\eta)p_3d_2 + (1-\eta)p_2d_3 + (1-\eta)p_1d_4 + d_5] + \cdots$$

By comparing the coefficient with respect to the power of ξ and Lemma 2.4, the inequality (9) is obtained. Equivalently

$$3d_3[3]_q\beta + 2\beta(\beta-1)d_2^2[2]_2^2 = (1-\eta)p_1 + (1-\eta)p_2d_2 + d_3$$

so that

$$([3]_q\beta - 1)d_3 = (1-\eta)\left[p_2 - \frac{(1-\eta)(2\beta^2 - 4\beta + 1)}{4\beta^2 - 4\beta + 1}\frac{p_1^2}{2} \right]$$

For $\beta \geq 1$, Lemma 2.2 and triangle inequality, the inequality (10) is obtained.
For d_4, we have

$$d_4 = \frac{1-\eta}{(\beta[4]_q - 1)} p_3 - \frac{(1-\eta)^2}{(\beta[4]_q - 1)} \left[\frac{6\beta^2 - 11\beta + 2}{(2\beta-1)(3\beta-1)} \right] p_1 p_2 + \frac{(1-\eta)^3}{(4\beta[4]_q - 1)} \left[\frac{24\beta^4 - 80\beta^3 + 84\beta^2 - 28\beta + 3}{3(2\beta-1)^3(3\beta-1)} \right] p_1^3.$$

By Lemma 2.2 and triangle inequality, the inequality (11) is obtained.
For d_5, we have

$$d_5 = \Gamma p_4 - \Upsilon p_1 p_3 - \Pi p_2^2 + \Psi p_1^2 p_2 - \Sigma p_1^4,$$

this can further simplify as

$$\Gamma \left[p_4 - \frac{\Upsilon}{\Gamma} p_1 p_3 \right] - \Pi p_2 \left[p_2 - \frac{\Psi}{\Pi} p_1^2 \right] - \Sigma p_1^4.$$

Taking the absolute value of d_5, applying triangle inequality with Lemmas 2.2 and 2.3, the inequality (12) is obtained. □

Corollary 3.7. *For $q \to 1$, the coefficient inequalities in [4] is obtained.*

Corollary 3.8. *For $q \to 1$, $\beta = 1$, the coefficient inequalities in [6] is obtained.*

Theorem 3.5. *For $\rho \in \mathbb{R}$, $f(\xi) \in \mathbf{S}_q(\eta, \beta)$. Then*

$$|d_3 - \rho d_2^2| \leq \begin{cases} \frac{2(1-\eta)(1-\Lambda)}{([3]_q\beta-1)}, \ for\ \rho \leq \frac{[2]_q^2(4\beta-2\beta^2-1)}{([3]_q\beta-1)} \\ \\ \frac{2(1-\eta)}{([3]_q\beta-1)}, \ for\ \frac{[2]_q^2(4\beta-2\beta^2-1)}{([3]_q\beta-1)} \leq \rho \leq \frac{[2]_q^2(4\beta-2\beta^2-1)}{([3]_q\beta-1)} + \frac{[2]_q^2(2\beta-1)^2}{(1-\eta)([3]_q\beta-1)} \\ \\ \frac{2(1-\eta)(1-\Lambda)}{([3]_q\beta-1)}, \ for\ \rho \geq \frac{[2]_q^2(4\beta-2\beta^2-1)}{([3]_q\beta-1)} + \frac{[2]_q^2(2\beta-1)^2}{(1-\eta)([3]_q\beta-1)} \end{cases} \tag{15}$$

Where

$$\Lambda = \frac{2(1-\eta)[[2]_q^2(2\beta^2 - 4\beta + 1) + \rho([3]_q\beta - 1)]}{[2]_q^q(2\beta-1)^2} \tag{16}$$

Proof. Since the functional is given as

$$|d_3 - \rho d_2^2| \tag{17}$$

Combining the value of d_3, d_2 in the equation (17), we have

$$|d_3 - \rho d_2^2| = \left| \frac{(1-\eta)}{([3]_q\beta - 1)} \left[p_2 - \frac{2(1-\eta)[[2]_q^2(2\beta^2 - 4\beta + 1) + \rho([3]_q\beta - 1)]}{[2]_q^2(2\beta-1)^2} \right] \right| \tag{18}$$

and

$$|d_3 - \rho d_2^2| = \frac{(1-\eta)}{([3]_q\beta - 1)} |p_2 - \Lambda \frac{p_1^2}{2}|.$$

By Lemma 2.2, the results (15) is obtained for the ranges of ρ, with Λ define in (16). □

Corollary 3.9. *For $q \to 1$, the coefficient inequalities in [4] is obtained.*

Corollary 3.10. *For $q \to 1$, $\beta = 1$, the coefficient inequalities in [6] is obtained.*

Theorem 3.6. *For $\chi \in \mathbb{C}$, $f(\xi) \in \mathbf{S}_q(\eta, \beta)$. Then*

$$\left|d_3 - \chi d_2^2\right| \leq \frac{2(1-\eta)}{([3]_q\beta - 1)} \max\{1, |1-\lambda|\} \tag{19}$$

Where

$$\lambda = \frac{2(1-\eta)[[2]_2^2(2\beta^2 - 4\beta + 1) + \chi([3]_q\beta - 1)]}{[2]_q^2(2\beta - 1)^2} \tag{20}$$

Proof. Since the functional is given as

$$|d_3 - \rho d_2^2| \tag{21}$$

Combining the value of d_3, d_2 in the equation (16), we have

$$|d_3 - \rho d_2^2| = \left|\frac{(1-\eta)}{([3]_q\beta - 1)}\left[p_2 - \frac{2(1-\eta)[[2]_q^2(2\beta^2 - 4\beta + 1) + \chi(3[3]_q\beta - 1)]}{[2]_q^2(2\beta - 1)^2}\right]\right| \tag{22}$$

and

$$|d_3 - \rho d_2^2| = \frac{(1-\eta)}{([3]_q\beta - 1)}|p_2 - \chi\frac{p_1^2}{2}|.$$

By Lemma 2.3, the results (19) is obtained. □

Corollary 3.11. *For $q \to 1$, the coefficient inequalities in [4] is obtained.*

Corollary 3.12. *For $q \to 1$, $\beta = 1$, the coefficient inequalities in [6, 12] is obtained.*

4. Conclusion

This study has given a new concept with the quantum approach for the class β-pseudo starlike functions introduced in [4] in the area geometry function theory which researchers in the field can further study by looking at some other properties of the class of the function.

The Hankel determinant for this class of functions is in progress by the authors of this work.

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