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Nano $(1,2)^{\star}$-locally closed sets and its generalization in nano bitopological spaces

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Abstract: In this paper, we focused on the $N_{(1,2)^{\star}}$-LC set in nanobitopological spaces and certain properties of these investigated. Also, we studied the $N_{(1,2)^{\star}}$-GLC set, $N_{(1,2)^{\star}}$-$GLC^{\star}$ set, and $N_{(1,2)^{\star}}$-$GLC^{\star\star}$ set and established their various characteristic properties.

Key words: $N_{(1,2)^{\star}}$-LC set, $N_{(1,2)^{\star}}$-GLC set, $N_{(1,2)^{\star}}$-$GLC^{\star}$ set and $N_{(1,2)^{\star}}$-$GLC^{\star\star}$ set .

1. Introduction and Preliminaries

In order to study the imperfect data in mathematics, Lellis Thivagar *et al*,[4], introduced a new topology, which overtook all of these theories. The new topology is known as Nano Topology due to its small size. Regardless of the universe size, a subset of a universe can be reduced to a maximum of five open sets by identifying its boundary regions and lower and upper approximations. The elements of the nanotopology are the nano open sets. Lellis Thivagar further established weak forms of nanotopology, nanotopology in $\check{C}$ech rough closure space, and other subjects *et al*, [5]. Buvaneshwari *et al* investigated the concept of nano $(1,2)^{\star}$-g-closed sets in [2]. In this study, we studied some characteristics of the $N_{(1,2)^{\star}}$-LC set in nano bitopological spaces. Likewise we analyzed and determined the various unique characteristics of the $N_{(1,2)^{\star}}$-LC set, $N_{(1,2)^{\star}}$-GLC set and $N_{(1,2)^{\star}}$-$GLC^{\star}$ set and so on. Furthermore, this work is motivated by some research work, for example ([1], [6], [7], [9], [10] and [11]).

Definition 1.1. [8] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U,R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.

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3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not $-X$ with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 1.2. [4] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$,
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,
3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

Definition 1.3. [3] Let U be the universe, R be an equivalence relation on U and $\tau_{R_{1,2}}(X) = \bigcup\{\tau_{R_1}(X), \tau_{R_2}(X)\}$ where $X \subseteq U$. Then it is satisfies the following axioms:

1. U and $\phi \in \tau_{R_{1,2}}(X)$.
2. The union of the elements of any sub collection of $\tau_{R_{1,2}}(X)$ is in $\tau_{R_{1,2}}(X)$.
3. The intersection of the elements of any finite sub collection of $\tau_{R_{1,2}}(X)$ is in $\tau_{R_{1,2}}(X)$.

Then $\tau_{R_{1,2}}(X)$ is a topology on U called the Nano bitopology on U with respect to X. $(U, \tau_{R_{1,2}}(X))$ is called the Nano bitopological space. Elements of the Nano bitopology are known as Nano $\tau_{1,2}$-open sets in U. Elements of $(\tau_{R_{1,2}}(X))^c$ are called Nano $\tau_{1,2}$-closed sets in $\tau_{R_{1,2}}(X)$.

Definition 1.4. [3] If $(U, \tau_{R_{1,2}}(X))$ is a Nano bitopological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then

1. The Nano $(1,2)^\star$ closure of A is defined as the intersection of all Nano $(1,2)^\star$ closed sets containing A and it is denoted by $N_{\tau_{1,2}}\text{-}cl(A)$.
2. The Nano $(1,2)^\star$ interior of A is defined as the union of all Nano $(1,2)^\star$ open subsets of A contained in A and it is denoted by $N_{\tau_{1,2}}\text{-}int(A)$.

Definition 1.5. [3] Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and $A \subseteq U$. Then A is said to be a Nano $(1,2)^\star$-semi open set if $A \subseteq N_{\tau_{1,2}}\text{-}cl(N_{\tau_{1,2}}\text{-}int(A))$. The complements of open set is called closed set.

Definition 1.6. [2] Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and $A \subseteq U$. Then A is said to be a nano $(1,2)^\star$-g-closed set (briefly, $N_{(1,2)^\star}$-g-closed set) if $N_{\tau_{1,2}}\text{-}cl(G) \subseteq V$ where $G \subseteq V$ and V is nano $\tau_{1,2}$-open. The complements of the closed set is called open set.

Example 1.1. *[12] A subset A of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$ is called a nano $(1,2)^\star$-dense if $N_{\tau_{1,2}}\text{-}cl(A) = U$.*

2. Nano $(1,2)^{\star}$-locally closed sets

Definition 2.1. A subset W of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$ is said to be a nano $(1,2)^{\star}$-locally closed set (short in $N_{(1,2)^{\star}}$-LC set) if $W = K \cap L$ where K is $N_{\tau_{1,2}}$-open and L is $N_{\tau_{1,2}}$-closed in $(U, \tau_{R_{1,2}}(X))$.

Example 2.1. *Let $U = \{y_1, y_2, y_3, y_4\}$ with $U/R_1 = \{\{y_1\}, \{y_2\}, \{y_3\}, \{y_4\}\}$ and $X = \{y_1\}$ then $\tau_{R_1}(X) = \{\phi, \{y_1\}, U\}$ and let $U/R_2 = \{\{y_1, y_4\}, \{y_2, y_3\}\}$ and $X = \{y_2, y_3\}$ then $\tau_{R_2}(X) = \{\phi, \{y_2, y_3\}, U\}$. Then the sets in $\{\phi, \{y_1\}, \{y_2, y_3\}, \{y_1, y_2, y_3\}, U\}$ are called $N_{\tau_{1,2}}$-open and the sets in $\{\phi, \{y_4\}, \{y_1, y_4\}, \{y_2, y_3, y_4\}, U\}$ are called $N_{\tau_{1,2}}$-closed. In the nano bitopological space $(U, \tau_{R_{1,2}}(X))$, then the subset $\{y_1\}$ is $N_{(1,2)^{\star}}$-LC set and $\{y_2\}$ is not $N_{(1,2)^{\star}}$-LC set.*

Definition 2.2. A nano bitopological space $(U, \tau_{R_{1,2}}(X))$ is called nano $(1,2)^{\star}$-T_1-space (briefly, $N_{(1,2)^{\star}}$-T_1-space) for $a, b \in U$ and $a \neq b$, there exists a $N_{\tau_{1,2}}$-open sets W and V such that $a \in W$, $b \notin W$ and $b \in V$, $a \notin V$.

Theorem 2.1. 1. *Let $(U, \tau_{R_{1,2}}(X))$ be a $N_{(1,2)^{\star}}$-T_1-space and let W be a discrete subset of $(U, \tau_{R_{1,2}}(X))$. Then W is $N_{(1,2)^{\star}}$-LC set.*

2. *Let $(U, \tau_{R_{1,2}}(X))$ be nano $(1,2)^{\star}$-dense in itself and W be a discrete subset. Then $U - W$ is $N_{(1,2)^{\star}}$-LC set if and only if W is $N_{\tau_{1,2}}$-closed.*

Proof. Let W be a discrete subset of the $N_{(1,2)^{\star}}$-T_1-space $(U, \tau_{R_{1,2}}(X))$, i.e,. for each $x \in W$ there is a $N_{\tau_{1,2}}$-open set E_U such that $E_U \cap W = x$. If $E = E_U | x \in W$ then it is easily verified that $W = E \cap N_{\tau_{1,2}}$-$cl(W)$. This proves (1).

In order to prove (2), observe that in a nano $(1,2)^{\star}$-dense in itself space any discrete subset has empty $N_{\tau_{1,2}}$-interior.

Theorem 2.2. *Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and let $X \in N_{(1,2)^{\star}}LC(U)$. If $W \in X$ and $W \in N_{(1,2)^{\star}}$-$LC(X, \mathcal{U}/X)$ then $W \in N_{(1,2)^{\star}}$-$LC(U)$.*

Theorem 2.3. *Let W and Z be $N_{(1,2)^{\star}}$-LC subsets of a space $(U, \tau_{R_{1,2}}(X))$. If W and Z are separated, i.e. if $W \cap N_{\tau_{1,2}}$-$cl(Z) = N_{\tau_{1,2}}$-$cl(W \cap Z) = \phi$, then $W \cup Z \in N_{(1,2)^{\star}}$-$LC(U)$.*

Proof. Suppose there are $N_{\tau_{1,2}}$-open sets P and S such that $W = P \cap N_{\tau_{1,2}}$-$cl(W)$ and $Z = S \cap N_{\tau_{1,2}}$-$cl(Z)$. Since W and Z are separated we may assume that $P \cap N_{\tau_{1,2}}$-$cl(Z) = S \cap N_{\tau_{1,2}}$-$cl(W) = \phi$. Consequently, $W \cup Z = P \cup S \cap N_{\tau_{1,2}}$-$cl((W) \cup Z$ verifies that $W \cup Z \in N_{(1,2)^{\star}}$-$LC(U)$.

Theorem 2.4. *Let $\{X_i | i \in I\}$ be either an $N_{\tau_{1,2}}$-open cover or a $N_{(1,2)^{\star}}$- locally finite closed cover of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$ and let $W \subseteq U$. If $W \cap X_i \in N_{(1,2)^{\star}}$-$LC(X_i, \mathcal{U}/X_i)$ for each $i \in I$ then $W \in N_{(1,2)^{\star}}$-$LC(U)$.*

Proof. First suppose that $\{X_i | i \in I\}$ is an $N_{\tau_{1,2}}$-open cover of $(U, \tau_{R_{1,2}}(X))$. For each $i \in I$, since $W \cap X_i \in N_{(1,2)^{\star}}$-$LC(X_i, \mathcal{U}/X_i)$ we may assume that $W \cap X_i = S_i \cap N_{\tau_{1,2}}$-$cl(W \cap X_i)$ where $S_i \in \mathcal{U}$ and $S_i \subseteq X_i$. Now $S_i \cap N_{\tau_{1,2}}$-$cl(W) = S_i \cap X_i \cap N_{\tau_{1,2}}$-$cl(W) \subseteq S_i \cap N_{\tau_{1,2}}$-$cl(W \cap X_i) = W \cap X_i$. Hence if $S = \bigcup\{S_i | i \in I\}$ we have $S \cap N_{\tau_{1,2}}$-$cl(W) = W$.

Now suppose that $\{X_i|i \in I\}$ is a $N_{(1,2)^\star}$-locally finite closed cover of $(U, \tau_{R_{1,2}}(X))$. For each $i \in I$, since $W \cap X_i \in N_{(1,2)^\star}$-$LC(X_i, \mathcal{U}/X_i)$ we have $W \cap X_i = S_i \cap N_{\tau_{1,2}}$-$cl(W \cap X_i)$ where $S_i \in \mathcal{U}$. Let $x \in W$. Since $\{X_i|i \in I\}$ is a $N_{(1,2)^\star}$-locally finite closed cover, hence a point-finite and closure-preserving cover, there is a finite subset $I_U \subseteq I$ such that $u \in X_i$ if $I \in I_U$ and $x \notin \bigcup\{X_i|i \in I - I_U\}$. Moreover, there is an $N_{\tau_{1,2}}$-open set P_u containing x such that $P_U \subseteq \bigcap\{S_i|i \in I_U\}$ and $P_U \cap (P\{X_i|i \in I - I_U\}) = \phi$.

If $P = \bigcup\{P_u|u \in W\}$ then clearly $W \subseteq P \cap N_{\tau_{1,2}}$-$cl(W)$. Let $k \in P \cap N_{\tau_{1,2}}$-$cl(W)$. Then $k \in P_U$ for some $x \in W$. Since $k \in N_{\tau_{1,2}}$-$cl(W) = \bigcup\{N_{\tau_{1,2}}$-$cl(W \cap X_i)|i \in I\}$ we have $k \in N_{\tau_{1,2}}$-$cl(W \cap X_j)$ for some $j \in I$. Hence $j \in I_U$ and $P_U \subseteq S_j$. Thus $k \in S_j \cap N_{\tau_{1,2}}$-$cl(W \cap X_j) = W \cap X_j \subseteq W$. It follows that $W = P \cap N_{\tau_{1,2}}$-$cl(W)$.

Theorem 2.5. *For a $N_{(1,2)^\star}$-T_1-space $(U, \tau_{R_{1,2}}(X))$ the following are equivalent:*

1. *$W \in N_{(1,2)^\star}$-$LC(U)$ if and only if $U - W \in N_{(1,2)^\star}$-$LC(U)$.*
2. *$N_{(1,2)^\star}$-$LC(U)$ is $N_{\tau_{1,2}}$-closed under finite unions.*
3. *The boundary of each $N_{\tau_{1,2}}$-open set is a discrete subset.*
4. *The boundary of each nano $(1,2)^\star$-semi open set is a discrete subset.*
5. *Every nano $(1,2)^\star$-semi open set is $N_{(1,2)^\star}$-LC.*

Proof. $(1) \Leftrightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$: Let P be $N_{\tau_{1,2}}$-open and let $x \in N_{(1,2)^\star}$-$d(P) \cap N_{\tau_{1,2}}$-$cl(P) \cap (U - P)$. By assumption, if $W = P \cup \{x\}$ then $W \in N_{(1,2)^\star}$-$LC(U)$. Let $W = S \cap N_{\tau_{1,2}}$-$cl(W)$ for some $N_{\tau_{1,2}}$-open set S. One easily verifies that $S \cap N_{(1,2)^\star}$-$d(P) = \{x\}$.

$(3) \Rightarrow (4)$: Let W be nano $(1,2)^\star$-semi open in $(U, \tau_{R_{1,2}}(X))$ and let $P = N_{\tau_{1,2}}$-$int(W)$. Then $N_{(1,2)^\star}$-$d(W) \subseteq N_{(1,2)^\star}$-$d(P)$ and hence $N_{(1,2)^\star}$-$d(W)$ is a discrete subset.

$(4) \Rightarrow (5)$: Let W be nano $(1,2)^\star$-semi open in $(U, \tau_{R_{1,2}}(X))$. For each $x \in W \cap N_{(1,2)^\star}$-$d(W)$ there is a $N_{\tau_{1,2}}$-open set P such that $P_U \cap N_{(1,2)^\star}$-$d(W) = \{x\}$. If $P = N_{\tau_{1,2}}$-$int(W) \cup [\bigcup\{P_U|x \in W \cap N_{(1,2)^\star}$-$d(W)\}]$ it is easily verified that $W = P \cap N_{\tau_{1,2}}$-$cl(W)$.

$(5) \Rightarrow (1)$: We will show that any union of a $N_{\tau_{1,2}}$-open set and a $N_{\tau_{1,2}}$-closed set is $N_{(1,2)^\star}$-LC. Let $C = E \cup G$ where E is $N_{\tau_{1,2}}$-open and G is $N_{\tau_{1,2}}$-closed. We may assume that $E \cap G = \phi$. If $W = E \cup N_{\tau_{1,2}}$-$(cl(E) \cap G)$ then $W)$ is nano $(1,2)^\star$-semi open and hence $W = S \cap N_{\tau_{1,2}}$-$cl(W) = S \cap N_{\tau_{1,2}}$-$cl(W) = S \cap N_{\tau_{1,2}}$-$cl(E)$ for some $N_{\tau_{1,2}}$-open set S. If $W = S \cup (U - N_{\tau_{1,2}}$-$cl(E))$ then clearly $C = W \cap N_{\tau_{1,2}}$-$cl(C)$. Thus $(C) \in N_{(1,2)^\star}$-$LC(U)$.

3. Nano $(1,2)^\star$ generalized locally closed sets

Definition 3.1. A subset W of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$, is called

1. a nano $(1,2)^\star$-generalized locally closed set (briefly, $N_{(1,2)^\star}$-GLC set) if $W = E \cap G$, where E is a $N_{(1,2)^\star}$-g-open set and G is a $N_{(1,2)^\star}$-g-closed set.
2. $N_{(1,2)^\star}$-$GLC^\star$ set if $W = E \cap G$, where E is a $N_{(1,2)^\star}$-g-open set and G is a $N_{\tau_{1,2}}$-closed set.

3. $N_{(1,2)^\star}$-$GLC^{\star\star}$ set if $W = E \cap G$, where E is a $N_{\tau_{1,2}}$-open set and G is a $N_{(1,2)^\star}$-g-closed set.

Proposition 3.1. *In a nano bitopological space $(U, \tau_{R_{1,2}}(X))$, every $N_{(1,2)^\star}$-g-closed set (resp. $N_{(1,2)^\star}$-g-open set) is $N_{(1,2)^\star}$-GLC.*

Remark 3.1. *The converse of Proposition 3.1 is need not be true in general as shown in the following example.*

Example 3.1. *In Example 2.1, then the subset $\{y_1\}$ is $N_{(1,2)^\star}$-GLC set but not $N_{(1,2)^\star}$-g-closed.*

Theorem 3.1. *For a subset W of $(U, \tau_{R_{1,2}}(X))$, the following statements are equivalent.*

1. *$W \in N_{(1,2)^\star}$-$GLC^\star(U)$.*
2. *$W = E \cap N_{\tau_{1,2}}$-$cl(W)$ for some $N_{(1,2)^\star}$-g-open set E.*
3. *$N_{\tau_{1,2}}$-$cl(W) - W$ is $N_{(1,2)^\star}$-g-closed.*
4. *$W \cup (P - N_{\tau_{1,2}}$-$cl(W))$ is $N_{(1,2)^\star}$-g-open.*

Proof. (1) $\Rightarrow$ (2): Let $W \in N_{(1,2)^\star}$-$GLC^\star(U)$. Then $W = E \cap G$ where E is $N_{(1,2)^\star}$-g-open and G is $N_{\tau_{1,2}}$-closed. Since $W \subseteq E$ and $W \subseteq N_{\tau_{1,2}}$-$cl(W)$, $W \subseteq E \cap N_{\tau_{1,2}}$-$cl(W)$.

Conversely, since $W \subseteq G$, $N_{\tau_{1,2}}$-$cl(W) \subseteq G$, we have $W = E \cap G$ contains $E \cap N_{\tau_{1,2}}$-$cl(W)$. That is $E \cap N_{\tau_{1,2}}$-$cl(W) \subseteq W$. Therefore we have $W = E \cap N_{\tau_{1,2}}$-$cl(W)$.

(2) $\Rightarrow$ (1): Since E is $N_{(1,2)^\star}$-g-open and $N_{\tau_{1,2}}$-$cl(W)$ is $N_{\tau_{1,2}}$-closed, $E \cap N_{\tau_{1,2}}$-$cl(W) \in N_{(1,2)^\star}$-$GLC^\star(U)$ by Definition of $N_{(1,2)^\star}$-$GLC^\star(U)$.

(2) $\Rightarrow$ (3): $W = E \cap N_{\tau_{1,2}}$-$cl(W)$ implies that $N_{\tau_{1,2}}$-$cl(W) - W = N_{\tau_{1,2}}$-$cl(W) \cap E^c$ which is $N_{(1,2)^\star}$-g-closed, Since E^c is $N_{(1,2)^\star}$-g-closed.

(3) $\Rightarrow$ (2): Let $E = N_{\tau_{1,2}}$-$cl(W) - W^c$. Then by assumption, E is $N_{(1,2)^\star}$-g-open in $(U, \tau_{R_{1,2}}(X))$ and $A = E \cap N_{\tau_{1,2}}$-$cl(W)$.

(3) $\Rightarrow$ (4): $W \cup P - N_{\tau_{1,2}}$-$cl(W) = W \cup (N_{\tau_{1,2}}$-$cl(W))^c = N_{\tau_{1,2}}$-$cl(W) - W^c$ and by assumption $(N_{\tau_{1,2}}$-$cl(W) - W)^c$ is $N_{(1,2)^\star}$-g-open and $W \cup P - N_{\tau_{1,2}}$-$cl(W)$ is $N_{(1,2)^\star}$-g-open.

(4) $\Rightarrow$ (3): Let $E = W \cup (N_{\tau_{1,2}}$-$cl(W))^c$. Then E^c is $N_{(1,2)^\star}$-g-closed and $E^c = N_{\tau_{1,2}}$-$cl(W) - W$ and therefore $N_{\tau_{1,2}}$-$cl(W) - W$ is $N_{(1,2)^\star}$-g-closed.

Theorem 3.2. *For a subset W of $(U, \tau_{R_{1,2}}(X))$, the following statements are equivalent.*

1. *$W \in N_{\tau_{1,2}}$-$GLC(U)$.*
2. *$W = E \cap N_{\tau_{1,2}}$-$gcl(W)$ for some $N_{(1,2)^\star}$-g-open set E.*
3. *$N_{\tau_{1,2}}$-$gcl(W) - W$ is $N_{(1,2)^\star}$-g-closed.*
4. *$W \cup (N_{\tau_{1,2}}$-$gcl(W))^c$ is $N_{(1,2)^\star}$-g-open.*
5. *$W \subseteq N_{\tau_{1,2}}$-$gint(W \cup (N_{\tau_{1,2}}$-$gcl(W)^c))$.*

Proof. (1) $\Rightarrow$ (2): Let $W \in N_{(1,2)^\star}\text{-}GLC(U)$. Then $W = (E, F) \cap G$, where E is $N_{(1,2)^\star}$-g-open and G is $N_{(1,2)^\star}$-g-closed. Since $W \subseteq G$, $N_{\tau_{1,2}}\text{-}gcl(W) \subseteq G$ and therefore $E \cap N_{\tau_{1,2}}\text{-}gcl(W) \subseteq W$. Also $W \subseteq E$ and $W \subseteq N_{\tau_{1,2}}\text{-}gcl(W)$ implies $W \subseteq E \cap N_{\tau_{1,2}}\text{-}gcl(W)$ and therefore $W = E \cap N_{\tau_{1,2}}\text{-}gcl(W)$.

(2) $\Rightarrow$ (3): $W = E \cap N_{\tau_{1,2}}\text{-}gcl(W)$ implies $N_{\tau_{1,2}}\text{-}gcl(W) - W = N_{\tau_{1,2}}\text{-}gcl(W) \cap E^c$ which is $N_{(1,2)^\star}$-g-closed since E^c is $N_{(1,2)^\star}$-g-closed.

(3) $\Rightarrow$ (4): $W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c = (N_{\tau_{1,2}}\text{-}gcl(W) - W)^c$ and by assumption $(N_{\tau_{1,2}}\text{-}gcl(W) - W)^c$ is $N_{(1,2)^\star}$-g-open and so is $W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c$.

(4) $\Rightarrow$ (5): By assumption, $W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c = N_{\tau_{1,2}}\text{-}gint(W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c)$ and hence $(W) \subseteq N_{\tau_{1,2}}\text{-}gint(W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c)$.

(5) $\Rightarrow$ (1): By assumption and since $W \subseteq N_{\tau_{1,2}}\text{-}gcl(W)$, $W = N_{\tau_{1,2}}\text{-}gint(W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c) \cap N_{\tau_{1,2}}\text{-}gcl(W) \in N_{(1,2)^\star}\text{-}GLC(U)$.

Theorem 3.3. *Let W be a subset of $(U, \tau_{R_{1,2}}(X))$. Then $W \in N_{(1,2)^\star}\text{-}GLC^{\star\star}(U)$ if and only if $W = E \cap N_{\tau_{1,2}}\text{-}gcl(W)$ for some $N_{\tau_{1,2}}$-open set E.*

Proof. Let $W \in N_{(1,2)^\star}\text{-}GLC^{\star\star}(U)$. Then $P = E \cap G$ where E is $N_{\tau_{1,2}}$-open and G is $N_{(1,2)^\star}$-g-closed. Since $P \subseteq G$, $N_{\tau_{1,2}}\text{-}gcl(W) \subseteq G$. Now $W = W \cap N_{\tau_{1,2}}\text{-}gcl(W) = E \cap G \cap N_{\tau_{1,2}}\text{-}gcl(W) = E \cap N_{\tau_{1,2}}\text{-}gcl(W)$. Here the converse part is trivial.

Corollary 3.1. *Let W be a subset of $(U, \tau_{R_{1,2}}(X))$. If $W \in N_{(1,2)^\star}\text{-}GLC^{\star\star}(U)$, then $N_{\tau_{1,2}}\text{-}gcl(W) - W$ is $N_{(1,2)^\star}$-g-closed and $W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c$ is $N_{(1,2)^\star}$-g-open.*

Proof. Let $W \in N_{\tau_{1,2}}\text{-}GLC^{\star\star}(U)$. Then by Theorem 3.3, $W = E \cap N_{\tau_{1,2}}\text{-}gcl(W)$ for some $N_{\tau_{1,2}}$-open set E and $N_{\tau_{1,2}}\text{-}gcl(W) - W = N_{\tau_{1,2}}\text{-}gcl(W) \cap E^c$ is $N_{(1,2)^\star}$-g-closed in $(U, \tau_{R_{1,2}}(X))$. If $G = N_{\tau_{1,2}}\text{-}gcl(W) - W$, then $G^c = W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c$ and G^c is $N_{(1,2)^\star}$-g-open and therefore $W \cup (N_{\tau_{1,2}}\text{-}gcl(W))^c$ is $N_{(1,2)^\star}$-g-open.

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