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The Baer-Kaplansky Property of Simple Modules

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Abstract: In this paper we study the Baer-Kaplansky property of simple modules. We show that if $R = F[x]$, where F is a subfield of an algebraically closed field C with $1 < [C : F] < \infty$, and $\mathcal{B}$ is a Baer-Kaplansky class of simple R-modules, then $|\mathcal{B}| \leq 2$. We also show that over many algebras (both hereditary and non-hereditary) and over many matrix rings (which are not algebras) the class of simple modules is not a Baer-Kaplansky class.

Key words: Baer-Kaplansky class, simple modules, quivers and representation.

1. Introduction

The celebrated Baer-Kaplansky theorem states that if A and C are torsion groups whose endomorphism rings are isomorphic, then every isomorphism Ψ between $\mathrm{End}_{\mathbb{Z}}(A)$ and $\mathrm{End}_{\mathbb{Z}}(C)$ is induced by a group isomorphism $\phi : A \longrightarrow C$ i.e., $\Psi : \eta \longmapsto \phi\eta\phi^{-1}$ (See [7, Theorem 108.1] or [12, Theorem 24.1]).

Motivated by this result, Ivanov and Vámos introduced the notion of *Baer-Kaplansky class* in [8]. According to [8], a class $\mathcal{C}$ of modules is called *Baer-Kaplansky*, if two of its modules are isomorphic whenever their endomorphism rings are isomorphic as rings.

By [10, Proposition 2.12], we know that the class of simple right R-modules is Baer-Kaplansky if and only if the class of semisimple right R-modules is Baer-Kaplansky, where R is any ring and recently it is proven in [11, Theorem 2.6] that, over a right semi-artinian ring R, if the class of simple right R-modules is Baer-Kaplansky, then the class of injective right R-modules is Baer-Kaplansky. As we see, the Baer-Kaplansky property of simple right R-modules is very close to the Baer-Kaplansky property of the other classes of modules such as semisimple modules or injective modules. Therefore, the aim of this paper is to investigate the Baer-Kaplansky property of the class of simple modules.

We first prove that if $R = F[x]$, where F is a subfield of an algebraically closed field C with $1 < [C : F] < \infty$, and $\mathcal{B}$ is a Baer-Kaplansky class of simple R-modules, then $|\mathcal{B}| \leq 2$ (Theorem 2.1). We also construct several examples of classes of simple modules satisfying or not satisfying the Baer-Kaplansky property. In Example 2.3, we construct a hereditary F-algebra R of finite representation type such that the class of simple right R-modules is not Baer-Kaplansky, and the class of indecomposable injective right R-modules is not

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Baer-Kaplansky. In Example 2.4 and Example 2.5, we construct two non-commutative and non-hereditary F-algebras of finite representation type such that the class of simple right modules is not Baer-Kaplansky, but the class of indecomposable injective right modules is Baer-Kaplansky. On the other hand, in Example 2.6, we construct a commutative F-algebra R such that the class of simple R-modules is not Baer-Kaplansky, but the class of finite dimensional (namely finitely generated) injective R-modules is Baer-Kaplansky. In Example 2.7, we prove that the class of simple modules over ring of Gaussian integers $\mathbb{Z}[i]$ is a Baer-Kaplansky class. Moreover, in Proposition 2.1, Proposition 2.2 and Corollary 2.1 we show that over many subrings of $\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{C} \end{pmatrix}$ the class of simple modules is not a Baer-Kaplansky class.

All rings are associative with identity and all modules are unital right modules. For a ring R and an R-module M, $\mathrm{End}_R(M)$ will denote the endomorphism ring of M. If X is any set, let $|X|$ denote the cardinal number of X. For any term not defined, the reader is referred to [1], [4] and [13].

2. Examples and Results

According to [10, Example 2.15] the class of simple R-modules is Baer-Kaplansky, where R is the commutative semisimple ring $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. On the other hand, the class of simple R-modules is not Baer-Kaplansky, where R is the commutative semisimple ring $\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and F is any field, by [10, Example 2.16]. Indeed, if $R = \mathbb{Z}_2 \oplus \mathbb{Z}_3$, then the simple R-modules are $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Since $|\mathrm{End}_R(\mathbb{Z}_2)| = 2$ and $|\mathrm{End}_R(\mathbb{Z}_3)| = 3$, it follows that $\{\mathbb{Z}_2, \mathbb{Z}_3\}$ is a Baer-Kaplansky class. And if $R = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ with F field, then the simple R-modules are $S_1 = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. Then $\dim_F(S_i) = 1$ for $i = 1, 2$ and so we have $\mathrm{End}_R(S_i) \subseteq \mathrm{End}_F(S_i) \cong F$. It follows that $\mathrm{End}_R(S_1) \cong F \cong \mathrm{End}_R(S_2)$. Hence $\{S_1, S_2\}$ is not a Baer-Kaplansky class.

Lemma 2.1. *Let $R = \mathbb{R}[x]$ and let $\mathcal{B}$ be a Baer-Kaplansky class of simple R-modules. Then $|\mathcal{B}| \leq 2$.*

Proof. Let S be a simple R-module. Then $S \cong R/(f)$ with f irreducible and monic. Now, one of the following cases occurs:

Case 1: If f has degree 1, then we have $\mathrm{End}_R(S) \cong \mathbb{R}$.

Case 2: If f has degree 2, then we have $\mathrm{End}_R(S) \cong \mathbb{C}$.

Hence $\mathcal{B}$ contains at most two simple modules. $\square$

We know from ([2], [3], [5, Theorem 3.1]and [9, Theorem 11.14]) that if C is an algebraically closed field and F is a subfield of C such that $1 < [C : F] < \infty$, then the characteristic of C is 0 and C is of the form $C = F(i)$ with $i^2 = -1$. Hence, if we always replace $\mathbb{R}$ by F and $\mathbb{C}$ by C in the proof of Lemma 2.1, we obtain the following result.

Theorem 2.1. *Let C be an algebraically closed field with a subfield F such that $1 < [C : F] < \infty$. Let $R = F[x]$ and let $\mathcal{B}$ be a Baer-Kaplansky class of simple R-modules. Then we have $|\mathcal{B}| \leq 2$.*

In the following we are giving more examples.

Example 2.1. *Let* $R = F[x]$ *with* F *field. Let* $S = R/(x)$ *and let* $T = R/(x-1)$. *Then we have* $S \ncong T$ *as* R*-modules and* $\dim_F(S) = 1 = \dim_F(T)$. *Consequently, we have* $\mathrm{End}_R(S) = \mathrm{End}_F(S) \cong F \cong \mathrm{End}_F(T) = \mathrm{End}_R(T)$. *Hence* $\{S, T\}$ *is not a Baer-Kaplansky class, and so the class of all simple* R*-modules is not Baer-Kaplansky.*

Example 2.2. *Let* X *be a subset of positive integers and let* F *be a field such that* $F[x]$ *contains an irreducible polynomial* f_n *of degree* n *for any* $n \in X$. *Then the class* $\{F[x]/(f_n) \mid n \in X\}$ *is a Baer-Kaplansky class. Indeed, for any* $n \in X$, *the endomorphism ring of simple module* $F[x]/(f_n)$ *has dimension* n.

For instance, let $X = \{1, 2, \ldots\}$. *If* $F = \mathbb{Q}$ *and* p *is a prime, then the polynomial* $f_n = x^n - p$ *is irreducible over* $\mathbb{Q}$ *for any* $n \in X$. *On the other hand, if* $F = \mathbb{Z}_p$ *for some prime* p *and* $L_n = \mathbb{Z}_p(a_n)$ *is an extension field of* $\mathbb{Z}_p$ *such that* $[L_n : \mathbb{Z}_p] = n$, *then the minimal polynomial* f_n *of* a_n *over* $\mathbb{Z}_p$ *has degree* n *and is irreducible over* $\mathbb{Z}_p$.

For the following two examples, see [6, Examples 2.1 and 2.3].

Example 2.3. *There is a hereditary* F*-algebra* R *of finite representation type such that the class of simple right* R*-modules is not Baer-Kaplansky. Indeed, let* R *be the* F*-algebra given by the quiver* $1 \to 3 \leftarrow 2$. *Consider the non-isomorphic simple right* R*-modules* 1 *and* 2. *Note that* $\mathrm{End}_R(1) \cong \mathrm{End}_R(2) \cong F$ *because* 1 *and* 2 *are one dimensional vector spaces. Therefore, the class of simple right* R*-modules is not a Baer-Kaplansky class. Since* 1 *and* 2 *are injective, it follows that the class of indecomposable injective modules is not Baer-Kaplansky, either.*

Example 2.4. *There is a non-hereditary* F*-algebra* R *of finite representation type such that the class of simple right* R*-modules is not Baer-Kaplansky. Indeed, let* R *be the* F*-algebra given by the quiver*

$$1 \xrightarrow{a} 2 \circlearrowright b$$

with relations $ab = b^2 = 0$. *Consider the non-isomorphic simple right* R*-modules* 1 *and* 2. *Then we have* $\mathrm{End}_R(1) \cong \mathrm{End}_R(2)$. *Therefore the class of simple right* R*-modules is not a Baer-Kaplansky class. Here the two indecomposable injective modules are* $I_1 = 1$ *and* $I_2 = \begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix}$. *Hence, we have* $\mathrm{End}_R(I_1) \cong F$ *and* $\mathrm{End}_R(I_2) \cong F[x]/(x^2)$, *and so* $\{I_1, I_2\}$ *is a Baer-Kaplansky class.*

As we shall see in Example 2.5, the opposite of the algebra considered in Example 2.4 has similar properties, but all its indecomposable injective modules are uniserial. We can compare the next examples with [11, Theorem 2.6].

Example 2.5. *Let* R *be the* F*-algebra given by the quiver*

$$b \circlearrowright 1 \xrightarrow{a} 2$$

with relations $ba = b^2 = 0$. *Then the simple right* R*-modules are the one dimensional modules* 1 *and* 2. *Hence we have that* $\mathrm{End}_R(1) \cong F \cong \mathrm{End}_R(2)$. *So, the class of simple right* R*-modules is not Baer-Kaplansky. On the other hand, the two indecomposable injective right* R*-modules are* $I_1 = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ *and* $I_2 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$. *Note that*

$\mathrm{End}_R(I_1) \cong F[x]/(x^2)$ *and* $\mathrm{End}_R(I_2) \cong F$. *Therefore, the class of indecomposable injective right R-modules is Baer-Kaplansky.*

Example 2.6. *Let R be the F-algebra given by the quiver*

$$x \circlearrowright \overset{\bullet}{1} \qquad\qquad \overset{\bullet}{2}$$

with relation $x^2 = 0$. Then the simple R-modules are 1 and 2 and so $\mathrm{End}_R(1) \cong F \cong \mathrm{End}_R(2)$. Hence the class of simple R-modules is not Baer-Kaplansky. On the other hand, the indecomposable injective R-modules are $I_1 = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ and $I_2 = 2$, and both of them are projective. Let L and M be two finite dimensional injective R-modules different from 0. Since R is commutative and noetherian, we may assume that $L = I_1^a \oplus I_2^b$ and $M = I_1^c \oplus I_2^d$ for some cardinals a, b, c, d by [14, Theorem 4.4]. Assume that $\mathrm{End}_R(L)$ and $\mathrm{End}_R(M)$ are isomorphic as rings. Since $\mathrm{End}_R(I_1) \cong F[x]/(x^2)$, $\mathrm{End}_R(I_2) \cong F$ and $\mathrm{Hom}_R(I_1, I_2) = \mathrm{Hom}_R(I_2, I_1) = 0$, *we have $\dim_F(\mathrm{End}_R(L)) = 2a^2 + b^2$ and $\dim_F(\mathrm{End}_R(M)) = 2c^2 + d^2$. We know from [15, Lemma 1] that $J(\mathrm{End}_R(L)) = \{f \in \mathrm{End}_R(L) \mid f(L) \text{ is small in } L\}$ and $J(\mathrm{End}_R(M)) = \{g \in \mathrm{End}_R(M) \mid g(M) \text{ is small in } M\}$. On the other hand, any small submodule of L (resp. of M) is contained in $\mathrm{Soc}(L) = \mathrm{Soc}(I_1^a) = 1^a$ (resp. in $\mathrm{Soc}(M) = \mathrm{Soc}(I_1^c) = 1^c$). Hence, we have $\dim_F(J(\mathrm{End}_R(L))) = a^2$ and $\dim_F(J(\mathrm{End}_R(M))) = c^2$. It follows that $a = c$, and hence $b = d$. Therefore, $M \cong L$. So, the class of finite dimensional injective right R-modules is Baer-Kaplansky.*

Next, we give an example of a commutative ring (which is neither finite, as $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, nor an algebra) such that the class of simple modules is Baer-Kaplansky.

Example 2.7. *Let R be the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. Let α be an irreducible element of R. Then one of the following cases occurs:*

Case 1: $\alpha = \pm p$ or $\alpha = \pm ip$ with p prime in $\mathbb{Z}$ and $p \equiv 3 \,(\mathrm{mod}\ 4)$. Then the simple module $R/(\alpha)$ is a field with p^2 elements. Hence, we have $|\mathrm{End}_R(R/(\alpha))| = p^2$.

Case 2: $\alpha = a + bi$ and $\alpha\bar{\alpha} = a^2 + b^2$ is a prime $p \in \mathbb{Z}$. Then we have $(p) \subsetneqq (\alpha)$ and so $R/(\alpha)$ is a field with p elements. Hence, we have $|\mathrm{End}_R(R/(\alpha))| = p$.

It follows that the class of simple modules is a Baer-Kaplansky class.

The next remark shows that $\mathbb{Z}[i]$ admits also a Baer-Kaplansky class of injective modules.

Remark 2.1. *For any positive prime $p \in \mathbb{Z}$, let H_p be the right $\mathbb{Z}[i]$-module $H_p = Hom_{\mathbb{Z}}(\mathbb{Z}[i], \mathbb{Z}(p^\infty))$ with $(hr)(s) = h(rs)$ for any $h \in H_p$ and $r, s \in \mathbb{Z}[i]$ [9, Proposition 3.4]. Since $\mathbb{Z}(p^\infty)$ is an injective $\mathbb{Z}$-module, we deduce from [9, Lemma 2, page 159] that H_p is an injective module over $\mathbb{Z}[i]$. Moreover, for any $x, y \in \mathbb{Z}(p^\infty)$, there is an element $h \in H_p$ such that $h(1) = x$ and $h(i) = y$ and we have $(hi)(1) = h(i) = y$ and $(hi)(i) = h(i^2) = h(-1) = -x$. Hence, the formula $(x, y)i = (y, -x)$ defines the structure of $\mathbb{Z}[i]$-module $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$, while the map $f : H_p \to \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ sending any $h \in H_p$ to $(h(1), h(i))$ becomes an isomorphism of $\mathbb{Z}[i]$-modules. We finally note that $End_{\mathbb{Z}[i]}(\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty))$ is a subring of the ring of all 2×2*

matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a,b,c,d \in J_p$, *where* J_p *is the ring of* p*-adic integers. Since* $(x,y)i = (y,-x)$ *for any* $(x,y) \in \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$, *the matrices describing an endomorphism of* $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ *have the property that* $(ay-cx, by-dx) = (y,-x)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ((x,y)i)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left((x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)i = (ax+cy, bx+dy)i =$ $(bx+dy, -ax-cy)$. *It follows that* $c=-b$ *and* $d=a$. *Thus, we have* $End_{\mathbb{Z}[i]}(\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a,b \in J_p \right\}$. *Since the additive group of* $\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a,b \in J_p \right\}$ *is isomorphic to* $J_p \oplus J_p$, *it follows that the class* $\{H_p \mid p \in \mathbb{N},\ p \text{ prime }\}$ *is a Baer-Kaplansky class of injective modules.*

The examples of the semisimple algebra $\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ suggest many examples of rings (which are neither semisimple nor algebras) such that the class of simple modules is not Baer-Kaplansky.

Proposition 2.1. *Let* a,p,q *be positive integers such that* p *and* q *are primes and* $a^2+p=q$. *Let* $\mathbb{Z}[\sqrt{p}i] = \{r + s\sqrt{p}i \mid r,s \in \mathbb{Z}\}$ *and let* $R = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z}[\sqrt{p}i] \end{pmatrix}$. *Then the class of simple* R*-modules is not Baer-Kaplansky.*

Proof. We first note that q is an irreducible element of $\mathbb{Z}$. Since $(a+\sqrt{p}i)(a-\sqrt{p}i) = a^2+p = q$, it follows that $a+\sqrt{p}i$ is an irreducible element of $\mathbb{Z}[\sqrt{p}i]$. Let $I_1 = \begin{pmatrix} (q) & 0 \\ 0 & \mathbb{Z}[\sqrt{p}i] \end{pmatrix}$ and let $I_2 = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & (a+\sqrt{p}i) \end{pmatrix}$. Then R/I_1 and R/I_2 are two non-isomorphic modules with q elements. Hence, we have $\mathrm{End}_R(R/I_1) \cong \mathbb{Z}_q \cong \mathrm{End}_R(R/I_2)$. □

Remark 2.2. *We list in the sequel the primes* $q \le 100$ *of the form* $q = a^2+p$ *with* p *prime:* $3 = 1^2+2$, $7 = 2^2+3$, $11 = 3^2+2$, $17 = 2^2+13$, $19 = 4^2+3$, $23 = 4^2+7$, $29 = 4^2+13$, $41 = 6^2+5$, $43 = 6^2+7$, $47 = 6^2+11$, $53 = 6^2+17$, $59 = 6^2+23$, $67 = 8^2+3$, $71 = 8^2+7$, $73 = 6^2+37$, $79 = 6^2+43$, $83 = 6^2+47$, $89 = 6^2+53$, $97 = 6^2+61$. *Consequently, the primes* ≤ 100 *which are not of the form* a^2+p *with* p *prime are* $2,5,13,31,37,61$.

Proposition 2.2. *Let* a *and* p *be positive integers such that* p *is prime and* $a^2-p=2$. *Let* R *be the ring* $\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z}[\sqrt{p}] \end{pmatrix}$, *where* $\mathbb{Z}[\sqrt{p}] = \{x + y\sqrt{p} \mid x,y \in \mathbb{Z}\}$. *Then* R *admits two non-isomorphic simple modules with two elements.*

Proof. Let $I_1 = \begin{pmatrix} (2) & 0 \\ 0 & \mathbb{Z}[\sqrt{p}] \end{pmatrix}$ and let Let $I_2 = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & (a+\sqrt{p}) \end{pmatrix}$. Then we have $2 = a^2-p = (a+\sqrt{p})(a-\sqrt{p})$. We also note that $\mathbb{Z}[\sqrt{p}]$ is the free abelian group generated by 1 and $\sqrt{p}$. It follows that the group $\mathbb{Z}[\sqrt{p}]/(a+\sqrt{p})$ is a vector space over $\mathbb{Z}_2$ generated by the vectors $v_1 = 1+(a+\sqrt{p})$ and $v_2 = \sqrt{p}+(a+\sqrt{p})$. Since $v_2 = \sqrt{p}+(a+\sqrt{p}) = \sqrt{p}-a-\sqrt{p}+(a+\sqrt{p}) = -[a+(a+\sqrt{p})] = -a[1+(a+\sqrt{p})] = -av_1$, we conclude that $\dim_{\mathbb{Z}_2}(\mathbb{Z}[\sqrt{p}]/(a+\sqrt{p})) = 1 = \dim_{\mathbb{Z}_2}(R/I_2)$. On the other hand, we clearly have $\dim_{\mathbb{Z}_2}(R/I_1) = 1$. Hence R/I_1 and R/I_2 are non-isomorphic simple R-modules with desired property. □

Remark 2.3. *This is the list of the pairs* (a, p) *satisfying the hypotheses of Proposition 2.2 with* $a < 50$: $(2,2)$, $(3,7)$, $(5,23)$, $(7,47)$, $(9,79)$, $(13,167)$, $(15,223)$, $(19,359)$, $(21,439)$, $(27,727)$, $(29,839)$, $(33,1087)$, $(35,1223)$, $(37,1367)$, $(43,1847)$, $(47,2207)$, $(49,2399)$.

Corollary 2.1. *Let* a, p *and* q *be positive integers with* p *and* q *primes such that* $a^2 - p = q$. *Then the ring* $\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z}[\sqrt{p}] \end{pmatrix}$ *admits two non-isomorphic simple modules with* q *elements.*

Proof. Repeat the proof of Proposition 2.2 with the following substitutions : $2 \mapsto q$, $(2) \mapsto (q)$ and $\mathbb{Z}_2 \mapsto \mathbb{Z}_q$. $\square$

Remark 2.4. *If* $q = 3$, *this is the list of the pairs* (a, p) *with* $a < 50$ *satisfying the hypotheses of Corollary 2.1 :* $(4,13)$, $(8,61)$, $(10,97)$, $(14,193)$, $(20,397)$, $(26,673)$, $(32,1021)$, $(34,1153)$, $(40,1597)$, $(44,1933)$, $(46,2113)$.

Conclusions

In this paper, we studied the class of simple right R-modules. We constructed several examples showing that the class of simple right R-modules is Baer-Kaplansky or not Baer-Kaplansky.

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References

[1] Anderson F.W. and Fuller K.R. Rings and Categories of Modules. Second Ed., Graduate Texts in Math. Springer, Berlin. 1992; Vol. 13.

[2] Artin E. and Schreier O. Algebraische Konstruktion reeller Körper. Artin's Collected papers. Springer-Verlag. 1965; 258-272.

[3] Artin E. and Schreier O. Eine Kennzeichnung der reell algeschlossenen Körper. Artin's Collected papers. Springer-Verlag. 1965; 289-295.

[4] Auslander M., Reiten I. and Smalø S.O. Representation Theory of Artin Algebras. Cambridge University Press. 1995.

[5] Conrad K. The Artin-Schreier theorem. Homepage of the author. Expository papers.

[6] D'Este G. and Keskin Tütüncü D. Baer-Kaplansky theorem for modules over non-commutative algebras. KYUNG-POOK Math. J. 2021; 63:213-222.

[7] Fuchs L. Infinite Abelian Groups. Pure and Applied Mathematics. New-York: Academic Press. 1973; Vol. 36-II.

[8] Ivanov G. and Vámos P. A characterization of FGC rings. Rocky Mountain J. Math. 2002; 32(4): 1485-1492.

[9] Jacobson N. Basic Algebra II. Freeman and Co. 1989.

[10] Keskin Tütüncü D. and Tribak R. On Baer-Kaplansky classes of modules. Algebra Colloq. 2017; 24:603-610.

[11] Keskin Tütüncü D. and Vedadi R. On the Baer-Kaplansky theorem for injective modules. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 2025; 119:1 (10 pages).

[12] Krylov A, Michalev A.V. and Tuganbaev A.A. Endomorphism Rings of Abelian Groups. Dordrecht: Springer, Kluwer Academic Publishers, 2003.

[13] Schiffler R. Quiver Representations. Springer International Publishing Switzerland. 2014.

[14] Sharpe D.W. and Vámos P. Injective Modules. Lectures in Pure Mathematics, Cambridge. 1972.

[15] Ware R. and Zelmanowitz J. The Jacobson radical of the endomorphism ring of a projective module. Proc. American Math. Soc. 1970; 26(1):15-20.