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Toroidal pseudo-differential operators and scalar quantization on Lie groups

Mykola Ivanovich Yaremenko[1*]
[1]National Technical University of Ukraine
"Igor Sikorsky Kyiv Polytechnic Institute" Kyiv, Ukraine
37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine, 03056.
0000-0002-9209-6059

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Abstract: Toroidal pseudo-differential operators on tori $T^n = R^n/Z^n$ are studied and global pseudo-differential calculus for symbols defined on $T^n \times Z^n$ is introduced on tori. We establish the condition on symbol associated with toroidal pseudo-differential operators under which toroidal pseudo-differential operators map in certain functional space, thus, assume $Op\,(\sigma)$ is the pseudo-differential operator associated with $\sigma\,(g,\ \xi)$ on $(T^n \times Z^n)$ that is continuous in g for each ξ and such that $\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma\,(g,\ \xi)\right| \le c\,(\alpha,\gamma)\,(1+|\xi|)^{-|\alpha|}$ then $Op\,(\sigma)$ extends to the bounded linear operator $L^p\,(T^n) \to L^p\,(T^n)$ for all $p \in (1,\ \infty)$. We consider a simply connected Lie group G with Haar measure μ, and g is Lie algebra associated with G, and assume a is a smooth function that satisfies the inequality $\int_{g'} \sup\limits_{X' \in g'} |\hat{a}\,(Z', X')|\, d\hat{\eta}\,(Z') < \infty$, then, the mapping $Op\,(a)$ is given by

$$Op\,(a)\,f\,(g) = \int_{g'}\int_G \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a\left(g, X'\right) f\,(h)\,d\mu\,(h)\,d\hat{\eta}\left(X'\right)$$

is a linear bounded operator $L^2\,(G) \to L^2\,(G)$.

Key words: periodic pseudodifferential operators, periodic integral operators, symbol analysts, Fourier transform, Lie group, Lie algebra, dynamic system, quantization.

1. Introduction

This article is dedicated to the theory of pseudo-differential operators on tori $T^n = R^n/Z^n$ with the symbols on $T^n \times Z^n$. Our goal is to investigate the general regularity properties of toroidal operators by elementary means employing methods of classical harmonic analysis. The symbols $\sigma\,(g,\ \xi)$ of toroidal pseudo-differential operators are defined on $T^n \times Z^n$ and continuous in g for each $\xi \in Z^n$, and satisfy the condition $\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma\,(g,\ \xi)\right| \le c\,(\alpha, s, \gamma)\,(1+|\xi|)^{s-\rho|\alpha|-\vartheta|\gamma|}$ for all $g \in T^n$, $\xi \in Z^n$ and all multi-indices $\alpha,\quad \gamma$, thus the classical restriction $\sigma \in C^\infty\,(T^n \times Z^n)$ can be mitigated.

The Fourier transform F maps $F\ :\ C^\infty\,(T^n) \to S\,(Z^n)$ is given by

$$F\,(\psi)\,(\xi) = \hat{\psi}\,(\xi) = \int_{T^n} \exp\,(-ig\cdot\xi)\,\psi\,(g)\,d\mu\,(g)$$

*Correspondence: math.kiev@gmail.com

with the normalizing measure μ, and its inverse Fourier transform $F^{-1} : S(Z^n) \to C^\infty(T^n)$ is

$$F^{-1}\left(\hat{\psi}\right)(g) = \sum_{\xi \in Z^n} \exp(ig \cdot \xi)\, \hat{\psi}(\xi) = \psi(g).$$

Applying this definition of Fourier transforms, we can give the most general definition of the toroidal pseudo-differential operator by the formula

$$A(\psi)(g) = \sum_{\xi \in Z^n} \exp(i2\pi g \cdot \xi)\, \sigma(g,\ \xi)\ \hat{\psi}(\xi).$$

The main idea of this kind of investigation is to establish relations between the classes of symbols and properties of the corresponding pseudo-differential operators, and the kernels of these operators.

Some aspects of the general theory and introduction of periodic operators can be found in [3], and further development [3, 9, 18] and [19], so in [9] authors establish the link between Weyl operators and Landau–Weyl calculus by the application of an infinite family of intertwining "windowed wavepacket transforms" that provides some clarity on regularity properties of Schrödinger equations. In [19], the Whittaker–Kotel'nikov–Shannon sampling theorem is investigated and shown that the sampling theorem is equivalent to the Valiron–Tschakaloff formula and the Paley–Wiener theorem. Using global representations by Fourier series, pseudo-differential and Fourier series operators on the torus is considered in [20], where some applications of the operator theory to the hyperbolic partial differential equations on tori are studied, so, the finite speed of propagation of singularities in hyperbolic problems permit cutting off the equation and initial date for large spatial variables for studying of solution properties for finite times, therefore, this problem can be embedded in the torus [20].

The results of this paper are especially interesting in view of the statement that the arbitrary, connected Abelian-Lie group is topologically isomorphic to $R^m \times T^n$ for some positive integers $m,\ n$, where $T^n = R^n/Z^n$ is m-torus. Thus, the global quantizations of toroidal pseudo-differential operators on such groups can be established. In the last chapter of the article, we consider the simple example of the Fourier transforms on a nilpotent Lie group with a Haar measure.

2. Toroidal pseudo-differential operators

We begin with the definition of a periodic pseudo-differential operator.

Definition 1. ***A periodic pseudo-differential operator with the toroidal symbol*** $\sigma \in C^\infty(T^n \times Z^n)$ ***is a continuous linear operator*** $A : C^\infty(T^n) \to C^\infty(T^n)$ ***given by***

$$\begin{aligned} &A_\sigma(\psi)(g) = \\ &= \textstyle\int_{Z^n} \exp(i2\pi g \cdot \xi)\, \sigma(g,\ \xi)\ \hat{\psi}(\xi)\ d\hat{\mu}(\xi) = \\ &= \textstyle\sum_{\xi \in Z^n} \exp(i2\pi g \cdot \xi)\, \sigma(g,\ \xi)\ \hat{\psi}(\xi) \end{aligned} \tag{1}$$

for all $\psi \in S(T^n)$ ***and*** $g \in T^n$***.***

The pseudo-differential operator $A : C^\infty(T^n) \to C^\infty(T^n)$ can be rewritten with the periodic Schwartz distributional kernel $K \in C^\infty(T^n \times T^n)$ in the form

$$A(\psi)(g) = \int_{T^n} K(g,\ h)\ \psi(h)\ d\mu(h) \tag{2}$$

or with the convolution kernel $k \in C^{\infty}(T^n \times T^n)$ in the form

$$A(\psi)(g) = \int_{T^n} k(g,\ g-h)\ \psi(h)\ d\mu(h) \tag{3}$$

so that $K(g,\ h) = k(g,\ g-h)$.

The kernel k of the linear operator $A_\sigma\ :\ C^{\infty}(T^n) \to C^{\infty}(T^n)$ is the inverse Fourier transform of the symbol $\sigma \in C^{\infty}(T^n \times Z^n)$ given by

$$\begin{array}{l} k(g,h) = \int_{Z^n} \exp(i2\pi h\cdot\xi)\,\sigma(g,\ \xi)\quad d\hat{\mu}(\xi) = \\ = \sum_{\xi\in Z^n} \exp(i2\pi h\cdot\xi)\,\sigma(g,\ \xi) = {F_{Z^n}}^{-1}(\sigma(g,\ \cdot))(h) \end{array} \tag{4}$$

and the symbol σ of the operator $A_\sigma\ :\ C^{\infty}(T^n) \to C^{\infty}(T^n)$ with the kernel $K(g,\ h) = k(g,\ g-h)$ is given by the Fourier transform

$$\sigma(g,\ \xi) = \int_{T^n} \exp(-i2\pi h\cdot\xi)\,k(g,\ h)\,d\mu(h) = F_{T^n}(k(g,\ \cdot))(\xi). \tag{5}$$

The forward and backward partial differences operators Δ_ξ^α and $\overline{\Delta}_\xi^\alpha$ are given by

$$\Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1}...\Delta_{\xi_n}^{\alpha_n}$$

$$\overline{\Delta}_\xi^\alpha = \overline{\Delta}_{\xi_1}^{\alpha_1}...\overline{\Delta}_{\xi_n}^{\alpha_n}$$

where $\Delta_{\xi_k}^{\alpha_k}\varphi(\xi) = \varphi(\xi+\delta_k) - \varphi(\xi)$ and $\overline{\Delta}_{\xi_k}^{\alpha_k}\varphi(\xi) = \varphi(\xi) - \varphi(\xi-\delta_k)$, here $(\delta_k)_j = \left\{ \begin{array}{lll} 1 & if & k=j \\ 0 & if & k\neq j \end{array} \right.$ for all $1 \le k, j \le n$, and for all differentiable functions $\varphi\ :\ Z^n \to C$.

Definition 2. ***A function*** $\sigma\ :\ T^n \times Z^n \to C$ ***is called a symbol of degree*** s ***and denoted*** $\sigma \in S_{\rho,\vartheta}^s(T^n \times Z^n)$ ***if the function*** $\sigma(g,\ \xi)$ ***is smooth in*** g ***for all elements*** $\xi \in Z^n$ ***and the inequality***

$$\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma(g,\ \xi)\right| \le c(\alpha,\gamma,s)\left(1+|\xi|^2\right)^{\frac{s-\rho|\alpha|-\vartheta|\gamma|}{2}} \tag{6}$$

for all multi-indices α ***and*** γ***, and for all*** $g \in T^n,\quad \xi \in Z^n$***.***

We define a function $N_{\rho,\vartheta}^s\ :\ S_{\rho,\vartheta}^s(T^n \times Z^n) \to R_+$ by

$$N_{\rho,\vartheta}^s(\sigma) = \sup_{(g,\ \xi)\in T^n\times Z^n} \left\{ \left(1+|\xi|^2\right)^{-\frac{s-\rho|\alpha|-\vartheta|\gamma|}{2}} \left|\Delta_\xi^\alpha \partial_g^\gamma \sigma(g,\ \xi)\right| \right\},$$

which determines a topology of the Frechet space on $S_{\rho,\vartheta}^s(T^n \times Z^n)$ by defining the countable family of seminorms $N_{\rho,\vartheta}^s$.

Definition 3. ***Let*** $\sigma \in S_{\rho,\vartheta}^s(T^n \times Z^n)$ ***then the operator given***

$$\psi(g) \mapsto A(\psi)(g) = \sum_{\xi\in Z^n} \exp(i2\pi g\cdot\xi)\,\sigma(g,\ \xi)\ \hat{\psi}(\xi) = Op(\sigma)(\psi)(g) \tag{7}$$

is called a toroidal pseudo-differential operator with the symbol $\sigma \in S^{s}_{\rho,\vartheta}\left(T^{n} \times Z^{n}\right)$***.***

The topology of $Op\left(S^{s}_{\rho,\vartheta}\left(T^{n} \times Z^{n}\right)\right)$ defines by the Frechet space topology of $S^{s}_{\rho,\vartheta}\left(T^{n} \times Z^{n}\right)$.

Expanding the Fourier transform and rewriting the toroidal pseudo-differential operator in the form

$$A_{\sigma}\left(\psi\right)\left(g\right) = \int_{T^{n}} \psi\left(h\right) \sum_{\xi \in Z^{n}} \exp\left(i2\pi\left(g-h\right)\cdot\xi\right)\sigma\left(g,\ \xi\right) d\mu\left(h\right), \tag{8}$$

we can define the Schwartz kernel $K\left(g,\ h\right)$ of the operator Op by

$$K\left(g,\ h\right) = \sum_{\xi \in Z^{n}} \sigma\left(g,\ \xi\right)\exp\left(i2\pi\left(g-h\right)\cdot\xi\right), \tag{9}$$

which is smooth everywhere with the possible exception of $g = h$.

Definition 4. ***For all multi-indices*** $\alpha,\ \gamma,\ \eta$***, and all*** $g,\ h \in T^{n}$ ***and*** $\xi \in Z^{n}$***, the*** C^{∞}***-continuous function*** $a\ :\ T^{n} \times T^{n} \times Z^{n} \to C$***, which satisfies the condition***

$$\left|\Delta^{\alpha}_{\xi}\partial^{\gamma}_{g}\partial^{\eta}_{h} a\left(g,\ h,\ \xi\right)\right| \le c\left(\alpha,\gamma,\eta,s\right)\left(1+\left|\xi\right|^{2}\right)^{\frac{s-\rho|\alpha|-\vartheta|\gamma+\eta|}{2}}, \tag{10}$$

is called the amplitude of the degree s***.***

We denote the set of all amplitudes $a\ :\ T^{n} \times T^{n} \times Z^{n} \to C$ ***of*** s***-degree by*** $A^{s}_{\rho,\vartheta}\left(T^{n} \times T^{n} \times Z^{n}\right)$***.***

For each amplitude $a \in A^{s}_{\rho,\vartheta}\left(T^{n} \times T^{n} \times Z^{n}\right)$, we define a linear toroidal operator

$$Op\left(a\right)\left(\psi\right)\left(g\right) = \sum_{\xi \in Z^{n}} \int_{T^{n}} \exp\left(i2\pi\left(g-h\right)\cdot\xi\right) a\left(g,\ h,\ \xi\right)\psi\left(h\right) d\mu\left(h\right)$$

for all functions $\psi \in C^{\infty}\left(T^{n}\right)$.

Theorem 1. ***Let*** $a \in A^{s}_{\rho,\vartheta}\left(T^{n} \times T^{n} \times Z^{n}\right)$ ***then the inequality***

$$\begin{gathered}\left|\Delta^{\alpha}_{\xi}\int_{T^{n}}\int_{T^{n}} \exp\left(i2\pi g\cdot\lambda\right)\exp\left(i2\pi h\cdot\tau\right) a\left(g,\ h,\ \xi\right) d\mu\left(h\right) d\mu\left(g\right)\right| \le \\ \le c\left(\alpha,\gamma,\eta,s\right)\left(1+\left|\lambda\right|^{2}\right)^{-\frac{\gamma}{2}}\left(1+\left|\tau\right|^{2}\right)^{-\frac{\eta}{2}}\left(1+\left|\xi\right|^{2}\right)^{\frac{s-\rho|\alpha|-\vartheta|\gamma+\eta|}{2}}\end{gathered} \tag{11}$$

holds for all $\lambda,\ \tau,\ \xi \in Z^{n}$ ***with the*** $c\left(\alpha,\gamma,\eta,s\right)$ ***depended only on*** α,γ,η,s***.***

Proof. By Lp_{g} we denote the Laplacian in g. By part integration, we have

$$\begin{aligned}&\left|\Delta^{\alpha}_{\xi}\int_{T^{n}}\int_{T^{n}} \exp\left(i2\pi g\cdot\lambda\right)\exp\left(i2\pi h\cdot\tau\right) a\left(g,\ h,\ \xi\right) d\mu\left(h\right) d\mu\left(g\right)\right| = \\ &= \left(1+\left|\lambda\right|^{2}\right)^{-\gamma}\left(1+\left|\tau\right|^{2}\right)^{-\eta}\times \\ &\left|\int_{T^{n}}\int_{T^{n}} \begin{matrix}\exp\left(i2\pi g\cdot\lambda\right)\exp\left(i2\pi h\cdot\tau\right)\left(1-\frac{1}{(2\pi)^{2}}Lp_{g}\right)^{\gamma}\times \\ \left(1-\frac{1}{(2\pi)^{2}}Lp_{h}\right)^{\eta}\Delta^{\alpha}_{\xi} a\left(g,\ h,\ \xi\right) d\mu\left(h\right) d\mu\left(g\right)\end{matrix}\right| \\ &\le \left(1+\left|\lambda\right|^{2}\right)^{-\gamma}\left(1+\left|\tau\right|^{2}\right)^{-\eta}\times \\ &\int_{T^{n}\times T^{n}}\left|\left(1-\frac{1}{(2\pi)^{2}}Lp_{g}\right)^{\gamma}\left(1-\frac{1}{(2\pi)^{2}}Lp_{h}\right)^{\eta}\Delta^{\alpha}_{\xi} a\left(g,\ h,\ \xi\right)\right| d\mu\left(g\times h\right),\end{aligned}$$

hence $a\in A^{s}_{\rho,\vartheta}\left(T^{n}\times T^{n}\times Z^{n}\right)$ the theorem is proven.

Theorem 2. ***Let*** $a\in C^{\infty}\left(T^{n}\times T^{n}\times Z^{n}\right)$ ***satisfies the inequality***

$$\left|\Delta^{\alpha}_{\xi}\partial^{\gamma}_{g}\partial^{\eta}_{h}a\left(g,\ h,\ \xi\right)\right|\leq c\left(\alpha,\gamma,\eta,s\right)\left(1+\left|\xi\right|^{2}\right)^{\frac{s}{2}}, \tag{12}$$

then the operator $Op\left(a\right)$ ***extends to the bounded operator*** $H^{m+s}\left(T^{n}\right)\to H^{m}\left(T^{n}\right)$ ***for all*** $m\in R$***.***

Proof. We calculate

$$\begin{aligned}&A\left(\psi\right)\left(g\right)=\\&=\textstyle\sum_{\xi\in Z^{n}}\int_{T^{n}}\exp\left(i2\pi\left(g-h\right)\cdot\xi\right)a\left(g,\ h,\ \xi\right)\ \psi\left(h\right)d\mu\left(h\right)=\\&=\textstyle\sum_{\xi,\lambda,\tau\in Z^{n}}\exp\left(i2\pi g\cdot\lambda\right)\hat{a}\left(\lambda-\xi,\ \xi-\tau,\ \xi\right)\ \hat{\psi}\left(\tau\right),\end{aligned}$$

so we denote

$$\widehat{A\left(\psi\right)}\left(\lambda\right)=\sum_{\xi,\tau\in Z^{n}}\hat{a}\left(\lambda-\xi,\ \xi-\tau,\ \xi\right)\ \hat{\psi}\left(\tau\right),$$

furthermore, we estimate the norm

$$\begin{aligned}&\left\|A\left(\psi\right)\right\|^{2}_{H^{m}\left(T^{n}\right)}=\textstyle\sum_{\lambda\in Z^{n}}\left(1+\left|\lambda\right|^{2}\right)^{m}\left|\widehat{A\left(\psi\right)}\left(\lambda\right)\right|^{2}\\&=\textstyle\sum_{\lambda\in Z^{n}}\left|\sum_{\xi,\tau\in Z^{n}}\left(1+\left|\lambda\right|^{2}\right)^{\frac{m}{2}}\hat{a}\left(\lambda-\xi,\ \xi-\tau,\ \xi\right)\ \hat{\psi}\left(\tau\right)\right|^{2}.\end{aligned}$$

Since

$$\left(1+\left|\lambda\right|^{2}\right)^{\frac{m}{2}}\leq$$

$$\leq2^{2m}\left(1+\left|\lambda-\xi\right|^{2}\right)^{\frac{m}{2}}\left(1+\left|\xi-\tau\right|^{2}\right)^{\frac{s+m}{2}}\left(1+\left|\tau\right|^{2}\right)^{\frac{s+m}{2}}\left(1+\left|\xi\right|^{2}\right)^{-\frac{m-s}{2}}$$

we have

$$\left(1+\left|\lambda\right|^{2}\right)^{\frac{m}{2}}\left|\hat{a}\left(\lambda-\xi,\ \xi-\tau,\ \xi\right)\right|\leq$$

$$\leq2^{2m}c\left(\gamma,\eta\right)\left(1+\left|\lambda-\xi\right|^{2}\right)^{\frac{m}{2}}\left(1+\left|\xi-\tau\right|^{2}\right)^{\frac{s+m}{2}}\left(1+\left|\tau\right|^{2}\right)^{\frac{s+m}{2}}$$

thus, we estimate

$$\begin{aligned}&\left\|A\left(\psi\right)\right\|^{2}_{H^{m}\left(T^{n}\right)}\leq\\&\leq c\textstyle\sum_{\lambda\in Z^{n}}\left(\sum_{\xi,\tau\in Z^{n}}\left(1+\left|\lambda-\xi\right|^{2}\right)^{\frac{m-\gamma}{2}}\right.\\&\left.\left(1+\left|\xi-\tau\right|^{2}\right)^{\frac{s+m-\eta}{2}}\left(1+\left|\tau\right|^{2}\right)^{\frac{s+m}{2}}\left|\hat{\psi}\left(\tau\right)\right|\right)^{2}\leq\\&\leq c\left(\sup_{\xi\in Z^{n}}\textstyle\sum_{\lambda\in Z^{n}}\left(1+\left|\lambda-\xi\right|^{2}\right)^{\frac{m-\gamma}{2}}\right)\\&\left(\textstyle\sum_{\xi,\tau\in Z^{n}}\left(1+\left|\xi-\tau\right|^{2}\right)^{\frac{s+m-\eta}{2}}\left(1+\left|\tau\right|^{2}\right)^{\frac{s+m}{2}}\left|\hat{\psi}\left(\tau\right)\right|\right)^{2}\leq\\&\leq c\left(\sup_{\xi\in Z^{n}}\textstyle\sum_{\lambda\in Z^{n}}\left(1+\left|\lambda\right|^{2}\right)^{\frac{m-\gamma}{2}}\right)\left(\textstyle\sum_{\xi\in Z^{n}}\left(1+\left|\xi\right|^{2}\right)^{\frac{s+m-\eta}{2}}\right)\left\|\psi\right\|^{2}_{H^{s+m}\left(T^{n}\right)}.\end{aligned}$$

Therefore, for large enough γ and η, we obtain $\|A(\psi)\|^2_{H^m(T^n)} \le c\|\psi\|^2_{H^{s+m}(T^n)}$. Theorem 2 is proven.

Thus, we have that assume a pseudo-differential operator $Op(\sigma)$ corresponds to the symbol $\sigma \in C^\infty(T^n \times Z^n)$ such that the inequality

$$\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma(g,\ \xi)\right| \le c(\alpha,\gamma,s)\left(1+|\xi|^2\right)^{\frac{s}{2}}$$

holds for all $g \in T^n$, $\xi \in Z^n$ and all multi-indices α, γ, then the operator $Op(\sigma)$ extends to the bounded operator $H^{m+s}(T^n) \to H^m(T^n)$ for all $m \in R$.

Theorem 3. ***If mapping*** $A\ :\ H^p(T^n) \to H^q(T^n)$ ***is a linear bounded operator for all*** $p,\ q \in R$***, then there exists a kernel*** $K \in C^\infty(T^n \times T^n)$ ***such that***

$$A(\psi)(g) = \int_{T^n} K(g,\ h)\,\psi(h)\,d\mu(h) \tag{13}$$

for all $\psi \in C^\infty(T^n)$***. The reverse is also true, the integral operator with the kernel*** $K \in C^\infty(T^n \times T^n)$ ***is the linear bounded operator*** $A\ :\ H^p(T^n) \to H^q(T^n)$ ***for all*** $p,\ q \in R$***.***

Proof. First, we have to show $\sigma \in \bigcap_{s\in R} S^s(T^n \times T^n)$, indeed, we estimate

$$\begin{aligned}
&\left|\partial_g^\gamma \sigma(g,\ \xi)\right| \le \textstyle\sum_{\chi\in Z^n} (2\pi)^{|\gamma|}\left(1+|\chi|^2\right)^{\frac{|\gamma|}{2}} |\widehat{\sigma}(\chi,\ \xi)| \le \\
&\le (2\pi)^{|\gamma|}\left(\textstyle\sum_{\chi\in Z^n}\left(1+|\chi|^2\right)^{-s}\right)^{\frac{1}{2}} \|g \mapsto \sigma(g,\ \xi)\|_{H^{|\gamma|+s}(T^n)} \\
&\le (2\pi)^{|\gamma|} 2^{|\gamma|+s}\left(\textstyle\sum_{\chi\in Z^n}\left(1+|\chi|^2\right)^{-s}\right)^{\frac{1}{2}}\left(1+|\chi|^2\right)^{\frac{|\gamma|+s+m}{2}} \|A\|_{LB\left(H^s(T^n),H^{|\gamma|+s}(T^n)\right)}
\end{aligned}$$

for each number $m \in R$, therefore $\sigma \in \bigcap_{s\in R} S^s(T^n \times T^n)$.

Second, we define the Schwartz kernel by

$$K(g,\ h) = \sum_{\xi\in Z^n} \sigma(g,\ \xi)\exp(i2\pi(g-h)\xi)$$

for $\sigma \in \bigcap_{s\in R} S^s(T^n \times T^n)$ so that $K \in C^\infty(T^n \times T^n)$ since

$$\partial_g^\alpha \partial_h^\eta K(g,\ h) = \sum_{\xi\in Z^n}(-i2\pi\xi)^\eta \sum_{\theta\le\alpha}\left(\begin{array}{c}\alpha\\ \theta\end{array}\right)\left(\partial_g^\theta \sigma(g,\ \xi)\right)\partial_g^{\alpha-\theta}\exp(i2\pi(g-h)\xi),$$

the series absolutely converges hence $\left|\partial_g^\alpha \sigma(g,\ \xi)\right| \le c(\alpha,l)\left(1+|\xi|^2\right)^{-\frac{l}{2}}$, that proves the direct statement of the theorem.

We obtain the reverse statement of the theorem if we take $a(g,\ h,\ \xi) = \delta_{0,\xi}K(g,\ h)$ where $\delta_{0,0} = 1$ and $\delta_{0,\xi} = 0$ for all $\xi \in Z^n$, $\xi \ne 0$.

For all $l \in R$, we have an estimation

$$\begin{aligned}
&\left|\Delta_\xi^\alpha \partial_g^\gamma \partial_h^\eta a(g,\ h,\ \xi)\right| \le 2^{|\alpha|}\left|\partial_g^\gamma \partial_h^\eta K(g,\ h)\right| 1([-|\alpha|,\ |\alpha|]^n) \le \\
&\le c(l,\alpha,\gamma,\eta)\left(1+|\xi|^2\right)^{-\frac{l}{2}}
\end{aligned}$$

where we denote the characteristic function of the set $[-|\alpha|,\ |\alpha|]^n \subset Z^n$ by $1([-|\alpha|,\ |\alpha|]^n)$. So, we obtain

$$A_a(\psi)(g) = \int_{T^n} \sum_{\xi \in Z^n} \exp(i2\pi(g-h)\cdot\xi)\,\sigma(g,\ \xi)\,\psi(h)\,d\mu(h) =$$
$$= \int_{T^n} K(g,\ h)\,d\mu(h) = A_K(\psi)(g).$$

Theorem 3 can be proven under rather milder restrictions on the symbol $\sigma(g,\ \xi)$ that it is continuous in g for each $\xi \in Z^n$ and satisfies the condition $\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma(g,\ \xi)\right| \le c(\alpha,\gamma)\left(1+|\xi|^2\right)^{-\frac{|\alpha|}{2}}$ for all $g \in T^n,\ \xi \in Z^n$ and all multi-indices $\alpha,\quad \gamma$.

3. Pseudo-differential operators on $L^p(T^n)$ and Sobolev spaces

Let the symbol $\sigma(g,\ \xi)$ on $(T^n \times Z^n)$ be continuous in g for each ξ and satisfies the condition

$$\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma(g,\ \xi)\right| \le c(\alpha,\gamma)(1+|\xi|)^{-|\alpha|} \tag{14}$$

for all $g \in T^n,\ \xi \in Z^n$ and all multi-indices $\alpha,\quad \gamma$. Then, there exists an $L^2(T^n)$-bounded operator $Op(\sigma)$ with the kernel K such that

$$Op(\sigma)(g) = \int_{T^n} K(g,\ h)\,\psi(h)\,d\mu(h), \tag{15}$$

where the kernel K satisfies the condition

$$\left|\partial_g^\alpha \partial_h^\gamma K(g,\ h)\right| \le c(\alpha,\gamma)\,|g-h|^{-n-|\alpha|-|\gamma|} \tag{16}$$

for all $g \neq h$ and for all multi-indices $\alpha,\quad \gamma$. Therefore, we have the theorem.

Theorem 4. ***Let*** $Op(\sigma)$ ***be the pseudo-differential operator with the symbol*** $\sigma(g,\xi)$ ***on*** $(T^n \times Z^n)$ ***be continuous in*** g ***for each*** ξ ***and satisfied*** (14) ***then*** $Op(\sigma)$ ***extends to the bounded linear operator*** $L^p(T^n) \to L^p(T^n)$ ***for all*** $p \in (1,\ \infty)$.

The **proof** of the theorem employs the Littlewood-Paley dyadic decomposition of the representation of the function as the composition of the functions with localized frequencies. So, assume fix a function $\eta \in C_0^\infty(T^n)$ and assume such that $\eta(\xi) = 1,\ |\xi| \le 1$ and $\eta(\xi) = 0,\ |\xi| \ge 2$; we define a function $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$. Then, the partitions of unity are given by $1 = \eta(\xi) + \sum_{k=1,2,\dots} \varphi(2^{-k}\xi) \quad \forall\xi,\ 1 = \sum_{k=-\infty,\ +\infty} \varphi(2^{-k}\xi) \quad \xi \neq 0$; the difference operator is $\Delta_k(\psi) = \psi * (\phi_{2^{-k}} - \phi_{2^{-k+1}})$, where $\phi_z(x) = z^{-n}\phi\left(\frac{x}{z}\right),\ \langle\phi\rangle = 1$ where inverse Fourier transform $\hat{\phi} = \eta$ for all $x \in T^n$. Assume ψ satisfies the Lipschitz conditions, so, there is a constant M such that inequality $\|\Delta_k(\psi)\|_{L^\infty} \le M2^{-kN}$ holds for the Lipschitz constant N.

The operator A_σ is decomposed into series $A_\sigma = \sum_{i=0,1,\dots} A_i$, where $A_i = A_\sigma \Delta_i$, and $A_\sigma \Delta_0 \psi = A_\sigma(\psi * \phi)$. Each operator A_n associates with a symbol $\sigma_i(x,\xi) = \sigma(x,\xi)\,\varphi(2^{-i}\xi)$ and $\sigma_0(x,\xi) = \sigma(x,\xi)\,\eta(\xi)$ for A_0. The difference operators satisfy the condition

$$\Delta_i = \Delta_i(\Delta_{i-1} + \Delta_i + \Delta_{i+1}).$$

We obtain $\mathrm{I} = \sum_{i=-\infty,\ +\infty} \Delta_n$. Then, the series $A_\sigma(\psi) = \sum_{i=0,1,\dots} A_i(\Delta_{i-1} + \Delta_i + \Delta_{i+1})\,\psi$, where $\|(\Delta_{i-1} + \Delta_i + \Delta_{i+1})\,\psi\|_{L^\infty} \le M2^{-kN}$, N is a Lipschitz coefficient. Since

$$\|\partial_x^\alpha A_i(\Delta_{i-1} + \Delta_i + \Delta_{i+1})\,\psi\|_{L^\infty} \le M_\sigma 2^{i(|\alpha|-N)},$$

so

$$\left\|\Delta_j \sum A_i \left(\Delta_{i-1} + \Delta_i + \Delta_{i+1}\right)\right\|_{L^\infty} \le M2^{-Nj}.$$

The kernel $k(x,t)$ is decomposed into the sum $\sum_{i=0,1,...} k_i(x,t)$ converging for each $x \in T^n$, we have the estimation $|\partial_x^\alpha \partial_t^\gamma k_i(x,t)| \le c(\alpha,\gamma)(M)|t|^{-M} 2^{i(n+|\gamma|-M)}$ since

$$(2\pi it)^\tau \partial_x^\alpha \partial_t^\gamma k_i(x,t) = \int_{\hat{G}} \partial_\chi^\tau (2\pi iz)^\gamma \partial_x^\alpha \sigma_i(x,\chi) \chi(t) d\hat{\mu}(\chi),$$

where index "i" in $\sigma_i(x,\cdot)$ means integer number, and in $2\pi it$, "i" means imaginary unit.

Next, let $|t| \ge 1$ and $M > n + |\gamma| - N$ then

$$\sum_{i=0,1,..} |\partial_x^\alpha \partial_t^\gamma k_i(x,t)| \le c(\alpha,\gamma)(M) O\left(|t|^{-M}\right)$$

that is less than $O\left(|t|^{-(n+|\gamma|-N)}\right)$ for arbitrary large N.

If $0 < |t| \le 1$, we divide the sum into two parts and estimate

$$\begin{aligned}&\textstyle\sum_{i=0,1,..} |\partial_x^\alpha \partial_t^\gamma k_i(x,t)| \le \\ &\le c(\alpha,\gamma)(M)|t|^{-M} \textstyle\sum_{2^i \le \frac{1}{|t|}} 2^{i(n+|\gamma|-M)} + \\ &+ c(\alpha,\gamma)(M)|t|^{-M} \textstyle\sum_{2^i > \frac{1}{|t|}} 2^{i(n+|\gamma|-M)},\end{aligned}$$

and we put $M = 0$ in the first sum and take $M > n + |\gamma|$ in the second sum, so that, we obtain

$$\sum_{i=0,1,..} |\partial_x^\alpha \partial_t^\gamma k_i(x,t)| \le O\left(|t|^{-(n+|\gamma|-N)}\right)$$

for all $0 < |t| \le 1$ and all N.

Thus, we have that if $\sigma \in C^\infty(T^n \times Z^n)$ satisfies the inequality

$$\left|\partial_h^\alpha \Delta_\chi^\gamma \sigma(h,\ \chi)\right| \le c(\alpha,\gamma)(1+|\chi|)^{-|\gamma|}$$

then the kernel $K \in C^\infty(T^n \times T)$ satisfies the inequality

$$|\partial_h^\alpha \partial_t^\gamma K(h,\ t)| \le c(\alpha,\gamma,N)|h-t|^{-n-|\gamma|-|\alpha|}$$

for all multi-indices α, γ and vectors $h \ne t$.

Theorem 5. ***Let*** $W_s^p(T^n)$ ***be a Sobolev space with the norm*** $\|\psi\|_{W_s^p(T^n)} = \sum_{|\gamma|\le s} \|\partial^\gamma \psi\|_{L^p(T^n)}$ ***and let*** $Op(\sigma)$ ***be the pseudo-differential operator with the symbol*** $\sigma(g,\ \xi)$ ***on*** $(T^n \times Z^n)$ ***be continuous in*** g ***for each*** ξ ***and satisfied***

$$\left|\Delta_\xi^\alpha \partial_g^\gamma \sigma(g,\ \xi)\right| \le c(\alpha,\gamma)(1+|\xi|)^{s-\alpha}$$

then $Op(\sigma)$ ***extends to the bounded linear operator*** $W_m^p(T^n) \to W_{m-s}^p(T^n)$ ***for all*** $m \geq s$ ***and*** $p \in (1, \infty)$**.**

The proof of the theorem follows from the previous theorem and the equality

$$\partial_g{}^{\gamma} A_{\sigma} = \sum_{|\alpha| \leq s} A_{\sigma_{\alpha}} \partial_g{}^{\alpha}$$

for $|\gamma| \leq m-s$ so that there exists the symbol σ_α of zero order, which corresponds to the operator A_{σ_α}. Thus, the operator $\partial_g{}^{\gamma} A_{\sigma}$ is an operator that corresponds to the symbol $\tilde{\sigma}$ satisfying the inequality

$$\left|\Delta_\xi^\alpha \partial_g^\gamma \tilde{\sigma}(g, \xi)\right| \leq c(\alpha, \gamma)(1+|\xi|)^{m-\alpha}.$$

4. Lie groups with the Haar measure

Let G be a simply connected Lie group that has an upper central series that terminates with itself. Let μ be a Haar measure on G. Let Lie algebra g be associated with the Lie group G. Then, for any function $f \in L^2(g)$, the Fourier transform is given by

$$F(f)(X') = \int_g \exp(-iX'(X)) f(X) \, d\eta(X),$$

where element X' belongs to dual g' to Lie algebra g, so that $X'(X) \in R$ for all $X \in g$ and $X' \in g'$. By incorporating exponential mapping into our considerations, we obtain $\tilde{F} = F \circ \exp$ given by

$$\tilde{F}(f)(X') = \int_G \exp(-iX'(\log(g))) \tilde{f}(g) \, d\mu(g),$$

where mapping $\log : G \to g$ is the inverse of the exponential $\exp : g \to G$. We remark that there are two kinds of exponential mappings: $\exp : g \to G$ is a diffeomorphism from g into G; and $\exp(-iX'(\log(g)))$ is a classical exponential function.

Reminding the classical Fourier transform $F : L^2(G) \to L^2\left(\hat{G}\right)$ given by

$$F(f)(\xi) = \hat{f}(\xi) = \int_G \overline{\xi(g)} f(g) \, d\mu(g)$$

and the inverse $F^{-1} : L^2\left(\hat{G}\right) \to L^2(G)$ by

$$F^{-1}(f)(g) = \int_{\hat{G}} \xi(g) f(\xi) \, d\hat{\mu}(\xi) = f(g),$$

we define the transformation T by

$$T(f)(X') = \int_G \int_{\hat{G}} \exp(-iX'(\log(g))) \xi(g) \hat{f}(\xi) \, d\hat{\mu}(\xi) \, d\mu(g),$$

which maps $T : L^2\left(\hat{G}\right) \to L^2(g')$. Similarly, we obtain a transformation $T : L^2\left(\hat{G}\right) \to L^2(g)$ defined by $T = \exp \circ F^{-1}$.

Definition 5. ***The pseudo-differential operator*** $Op(a) : L^2(G) \to L^2(G)$ ***is defined by***

$$\begin{aligned}&Op(a) f(g) = A(f)(g) = \\ &= \int_{g'} \int_G \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a(g, X') f(h) \, d\mu(h) \, d\hat{\eta}(X')\end{aligned}$$

for all $g \in G$***.***

The kernel K of the operator $Op(a)$ is the Fourier transform of the symbol a defined on $G \times g'$ given by

$$K(g,h) = \int_{g'} \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a(g, X') \, d\hat{\eta}(X'),$$

so that we obtain the standard view of the operator $Op(a)$ as

$$Op(a) f(g) = A(f)(g) = \int_G K(g,h) f(h) \, d\mu(h).$$

We rewrite $Op(a) : L^2(G) \to L^2(G)$ in the form

$$\begin{aligned}&A(f)(g) = \\ &= \int_{g'} \int_G \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a(g, X') f(h) \, d\mu(h) \, d\hat{\eta}(X') = \\ &= \int_G \int_{g'} \int_{g'} \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) \exp\left(iZ'\left(\log(g)\right)\right) \hat{a}(Z', X') f(h) \, d\hat{\eta}(Z') \, d\hat{\eta}(X') \, d\mu(h) = \\ &= \int_{g'} \tilde{A}^{Z'}(f)(g) \, d\hat{\eta}(Z')\end{aligned}$$

where we denote $\tilde{A}^{Z'}$ a linear operator given by

$$\begin{aligned}&\tilde{A}^{Z'}(f)(g) = \\ &= \exp\left(iZ'\left(\log(g)\right)\right) \int_G \int_{g'} \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) \hat{a}(Z', X') f(h) \, d\hat{\eta}(X') \, d\mu(h).\end{aligned}$$

We calculate the Fourier transform of the symbol a as

$$a(g, X') = \int_{g'} \exp\left(iZ'\left(\log(g)\right)\right) \hat{a}(Z', X') \, d\hat{\eta}(X')$$

and the inverse is given by

$$\hat{a}(Z', X') = \int_{g'} \exp\left(-iZ'\left(\log(h)\right)\right) a(h, X') \, d\mu(h).$$

By Plancherel theorem, we estimate

$$\left\| \tilde{A}^{Z'}(f) \right\|_{L^2} \le \sup_{X' \in g'} \left| \hat{a}(Z', X') \right| \|f\|_{L^2}$$

for all $Z' \in g'$ and all $f \in L^2$. So, if we demand

$$\int_{g'} \sup_{X' \in g'} \left| \hat{a}(Z', X') \right| d\hat{\eta}(Z') = M < \infty$$

then $\left\| \tilde{A}^{Z'} \right\|_{L^2} \le M$, thus, A_a is a bounded linear operator $L^2(G) \to L^2(G)$.

Thus, we proved Theorem 6.

Theorem 6. ***Let*** G ***be a simply connected Lie group with Haar measure*** μ***. Let*** g ***be Lie algebra associated with*** G***. Let the symbol*** a ***be a smooth function that satisfies the inequality***

$$\int_{g'} \sup_{X' \in g'} |\hat{a}(Z', X')| \, d\hat{\eta}(Z') < \infty.$$

Then, the mapping $Op(a)$ ***given by the formula***

$$Op(a) f(g) = \int_{g'} \int_G \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a(g, X') f(h) \, d\mu(h) \, d\hat{\eta}(X')$$

is a linear bounded operator $L^2(G) \to L^2(G)$***.***

5. Scalar quantization on Lie groups

First, we consider toroidal quantization. So, let the toroidal linear continuous operator $A : C^{\infty}(T^n) \to C^{\infty}(T^n)$ be associated with the toroidal symbol $\sigma \in C^{\infty}(T^n \times Z^n)$ then the kernel of the operator A is the Fourier transform of the symbol σ given by

$$K(g, \ h) = \sum_{\xi \in Z^n} \exp(i2\pi h \cdot \xi) \, \sigma(g, \ \xi),$$

thus, the quantization theorem states that the identity

$$A(\psi)(g) = \sum_{\xi \in Z^n} \exp(i2\pi g \cdot \xi) \, \sigma(g, \ \xi) \, \hat{\psi}(\xi)$$

holds for all $\psi \in C^{\infty}(T^n)$ and all $g \in T^n$.

Second, let G be a simply connected Lie group with the measure μ, and g' be dual to Lie algebra g associated with G. Let $\tau : G \to G$ be a continuous function.

The τ-pseudo-differential operator A_τ associated with the symbol a defined over $G \times g'$ is given by

$$\begin{aligned} &A_\tau(f)(g) = \\ &= \int_{g'} \int_G \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a\left(g\tau\left(gh^{-1}\right)^{-1}, X'\right) f(h) \, d\mu(h) \, d\hat{\eta}(X') \end{aligned}$$

for all $g \in G$ We can demand that operator A_τ can be represented by an integral with the distributional kernel K_τ, in the form

$$A_\tau(f)(g) = \int_G K_\tau(g, \ h) f(h) \, d\mu(h)$$

with the kernel $K_\tau : G \times G \to C$ defined by

$$K_\tau(g, \ h) = \int_{g'} \exp\left(iX'\left(\log\left(gh^{-1}\right)\right)\right) a\left(g\tau\left(gh^{-1}\right)^{-1}, X'\right) d\hat{\eta}(X')$$

for all $g, \ h \in G$.

The Weyl operator $W_\tau\left(\tilde{g},\tilde{X'}\right)\ :\ L^2(G)\to L^2(G)$ in $\left(\tilde{g},\tilde{X'}\right)\in G\times g'$ is given by

$$W_\tau\left(\tilde{g},\tilde{X'}\right)(f)(g)=\exp\left(iX'\left(\log\left(g\tau(\tilde{g})^{-1}\right)\right)\right)f\left(g\tilde{g}^{-1}\right).$$

The Fourier transform of the symbol a is given by

$$\hat{a}_{g'\times G}(X',g)=\int_{g'}\int_G\exp\left(-iX'(\log(h))\right)\exp\left(iZ'(\log(g))\right)a(h,Z')\,d\mu(h)\,d\hat{\eta}(Z')$$

for all $g,\in G$ and $X'\in g'$. The τ-pseudo-differential operator can be rewritten in the form

$$A_\tau(f)(g)=\int_{g'}\int_G\hat{a}_{g'\times G}(Z',h)\ W_\tau(h,Z')(f)(g)\,d\mu(h)\,d\hat{\eta}(Z'),$$

thus, there is the Weyl correspondence between symbols a and τ-pseudo-differential operators A_τ, this correspondence is called scalar τ-quantization on Lie groups.

The superposition of the Weyl operators W_τ is a Weyl operator so that

$$W_\tau\left(\tilde{g},\tilde{X'}\right)\ W_\tau\left(\tilde{h},\tilde{Y'}\right)=\Theta_\tau\left(\tilde{g},\tilde{X'};\tilde{h},\tilde{Y'}\right)\ W_\tau\left(\tilde{g}\tilde{h},\tilde{X'}+\tilde{Y'}\right)$$

here Θ_τ is the multiplication operator by

$$\Theta_\tau\left(\tilde{g},\tilde{X'};\tilde{h},\tilde{Y'}\right)=\exp\left(i\left(\begin{array}{l}\tilde{X'}\left(\log\left(\tilde{s}\tau(\tilde{g})^{-1}\right)-\log\left(\tilde{s}\tau\left(\tilde{g}\tilde{h}\right)^{-1}\right)\right)\\ -\tilde{Y'}\left(\log\left(\tilde{s}\tau\left(\tilde{g}\tilde{h}\right)^{-1}\right)-\log\left(\tilde{s}\tau\left(\tilde{h}^{-1}\right)\tilde{g}^{-1}\right)\right)\end{array}\right)\right).$$

For all $f\in L^2(G)$ and $\varphi\in L^2(G)$, we define a sesquilinear form by

$$\Psi_{a,\tau}(f,\varphi)=\langle A_\tau(f)\,,\ \varphi\rangle_{L^2(G)}.$$

Applying the Plancherel theorem, we estimate the sesquilinear form as

$$|\Psi_{a,\tau}(f,\varphi)|\le\|a\|_{L^2}\|f\|_{L^2}\|\varphi\|_{L^2}$$

for all $f,\ \varphi\in L^2(G)$.

Thus, we have theorem 7.

Theorem 7. ***If mapping*** B ***is a trace class operator from*** $L^2(G)$ ***to*** $L^2(G)$***, then there exists a sequence of complex numbers*** $\{\lambda_k\}\subset C$ ***such that*** $\sum_{k\in N}|\lambda_k|<\infty$***, and two orthonormal systems of functions*** $\{\theta_k\}\subset L^2(G)$ ***and*** $\{\vartheta_k\}\subset L^2(G)$ ***such that*** $B=\sum_{k\in N}\lambda_k\Psi_{a,\tau}(\theta_k,\vartheta_k)$***.***

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