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# A Fixed Point Theorem for Expansion onto Mappings on CCM-Spaces

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**Abstract:** In the present paper, we have obtained a fixed point theorem for expansion self mappings on CCM-spaces (Complete Cone Metric). Our result is an expansion and generalization of some of the results existing in the literature.

**Key words:** Cone metric space, fixed point, normal cones.
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## 1. Introduction

The concept of cone metric space was introduced by the authors Huang and Zhang [6], and they generalized the concept of a metric space into a cone metric space and replacing the real numbers by an ordered Banach space, also proved some of the fixed point theorems in cone metric spaces. Subsequently many authors has been inspiring with these results and generalizing, extending, improving, and modifying these results in many ways. ( for e.g., see, 1-5, 7-15). Some of the authors worked on fixed points for non expansive maps in cone metric spaces[1,14]. Recently Aage and Salunke [1] obtained some fixed point results in for non expansive maps in cone metric spaces. In the present paper , we have generalized results of Aage, and Salunke [1] and obtained results.

### 1.1. Preliminaries

The following are useful in our main results which are due to [14] .

**Definition 1.1.** Let $N$ be a real Banach space. A subset $F$ of $N$ is said to be a cone if and only if
(i) $F$ is closed, non-empty and $F \neq \{0\}$;
(ii) $\alpha, \beta \in R$ ,$\alpha, \beta \geq 0$, $u, v \in F$ implies $\alpha u + \beta v \in F$;
(iii) $F \cap (-F) = \{0\}$.
Given a cone $Fn \subset N$, we define a partial ordering $\leq$ with respect to $F$ by $\alpha \leq \beta$ iff $\beta - \alpha \in F$.
A cone $F$ is called normal if there exists a number $B > 0$ such that for all $\alpha, \beta \in F$ ,
$0 \leq \alpha \leq \beta \Rightarrow \|\alpha\| \leq B\|\beta\|$ .
The least positive number $B$ satisfying the above inequality is called the normal constant of $F$, while
$\alpha \ll \beta$ stands for $\beta - \alpha \in interior\ of F$.

**Definition 1.2.** Let $X$ be a non empty set of $N$ .Suppose that the map $\rho : X \times X \longrightarrow N$ satisfies :

(1) $0 \leq \rho(\alpha, \beta)$ for all $\alpha, \beta \in X$ and $\rho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;

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(2) $\rho(\alpha, \beta) = \rho(\beta, \alpha)$ for all $\alpha, \beta \in X$;

(3) $\rho(\alpha, \beta) \leq \rho(\alpha, \gamma) + \rho(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in X$.

Then $\rho$ is called a cone metric on $X$ and $(X, \rho)$ is called a CCM-Space(Cone Metric Space).

**Definition 1.3.** Let $(X, \rho)$ be a CCM-space.We say that $\{x_n\}$ is

$(i)$ a Cauchy sequence if for every $h$ in $N$ with $0 \ll c$, there is $N$ such that for all $n, m > N$, $\rho(x_n, x_m) \ll c$;

$(ii)$ a convergent sequence if for any $0 \ll h$, there is an $N$ such that for all $n > N$, $\rho(x_n, x) \ll h$, for some fixed $x$ in $X$. We denote this $x_n \longrightarrow x$ $(n \longrightarrow \infty)$.

**Definition 1.4.** A CCM-Space(Cone Metric Space) $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

## 2. Main Results

Now we prove our main theorem.

**Theorem 2.1.** *Let $(X, \rho)$ be a CCM-Space(Cone Metric Space) and a mapping $A : X \longrightarrow X$ be continuous onto mappings and satisfies the following condition:*

$$\rho(Ax, Ay) \geq \alpha_1 \rho(x, y) + \alpha_2 \rho(x, Ax) + \alpha_3 \rho(y, Ay) + \alpha_4 \rho(x, Ay) + \alpha_5 \rho(Ax, y). \tag{1}$$

*For all $x, y \in X$, where, $\alpha_1, \alpha_2 \geq -1$ and $\alpha_4 > 2, \alpha_3, \alpha_5 > 0$ are constants with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 > 1$.Then $A$ has a fixed point in $X$.*

*Proof.* For each $x_0 \in X$. $A$ is onto there exists $x_1 \in X$ such that $Ax_1 = x_0$. Similarly For each $n \geq 1$ there exists $x_{n+1} \in X$ such that $x_n = Ax_{n+1}$. If $x_{n-1} = x_n$, then $x_n$ is a fixed point of $A$. Thus suppose that $x_{n-1} \neq x_n$ for all $n \geq 1$. Then

$$\begin{aligned}
\rho(x_n, x_{n-1}) =& \rho(Ax_{n+1}, Ax_n) \\
\geq& \alpha_1 \rho(x_{n+1}, x_n) + \alpha_2 \rho(x_{n+1}, Ax_{n+1}) + \alpha_3 \rho(x_{n+1}, Ax_{n+1}) + \alpha_4 \rho(x_{n+1}, Ax_n) + \alpha_5 \rho(Ax_{n+1}, x_n), \\
=& \alpha_1 \rho(x_{n+1}, x_n) + \alpha_2 \rho(x_{n+1}, x_n) + \alpha_3 \rho(x_n, x_{n+1}) + \alpha_4 \rho(x_{n+1}, x_{n-1}) + \alpha_5 \rho(x_n, x_n), \\
\geq& (\alpha_1 + \alpha_2 + \alpha_4) \rho(x_{n+1}, x_n) + (\alpha_3 + \alpha_4) \rho(x_{n-1}, x_n), \\
\rho(x_n, x_{n-1}) \leq& (1 - \alpha_3 - \alpha_4)/(\alpha_1 + \alpha_2 + \alpha_4) \rho(x_{n-1}, x_n), \\
\leq& S \rho(x_{n-1}, x_n),
\end{aligned}$$

Where , $S = (1 - \alpha_3 - \alpha_4)/(\alpha_1 + \alpha_2 + \alpha_4) < 1$, with $0 < S < 1$ . Now for $n < m$ we have

$$\begin{aligned}
\rho(x_n, x_m) \leq& \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + ... + \rho(x_{m-1}, x_m), \\
\leq& (S^n + S^{n+1} + ... + S^{m-1}) \rho(x_0, x_1).
\end{aligned}$$

Let $0 \leq 1$ be given. Choose a natural number $K$ such that $(S^n/1 - S) \rho(x_0, x_1) < 1$, for all $n \geq K$. Then

$\rho(x_n, x_m) \leq 1$, for $n, m$. Therefore, $\{x_n\}$ is a Cauchy sequence in $(x, \rho)$. Since $(x, \rho)$ is a complete cone metric space, there exists $x_1 \in X$ such that $x_n \longrightarrow x_1$. If $A$ is continuous then

$\rho(Ax_1, x_1) \leq \rho(Ax_n, Ax_1) + \rho(Ax_n, x_1) \longrightarrow 0$ as $n \longrightarrow \infty$. Since $x_n \to x_1$ and $Ax_n \to Ax_1$ as $n \to \infty$. Therefore $\rho(Ax_1, x_1) = 0$.

Implies that $Ax_1 = x_1$. And $A$ has a fixed point in $X$. □

## 3. Conclusion

Our results are more general than the results of [1].

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