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Study on Extended B-K Fixed Point Theorem on CGM-Space Depended on an Another Function

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Abstract: In this paper we obtain, a fixed point theorem on extended B-K(Banach - Kannan) fixed point theorem on CGM-Space(Complete Generalized Metric) depends on another function. Our results are generalized, extended and improved results of some of the existing results in this literature.

Key words: contractive mapping, fixed point, generalized metric space, sequentially convergent, sub sequentially convergent.
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1. Introduction

In a non- linear analysis fixed point theory is one of the important topic. We considered The Banch fixed point theorem is first fixed point theorem. After that subsequently many authors (see for e.g. 1-5, 6-12) were extended and improved in different ways. In 1968, Kannan [4] obtained a fixed point theorem which is the most important theorem for further extension. And in 2000, Branciari [3] introduced a class of generalized metric spaces and prove some theorems. Recently S. Moradi[5] established a Kannan fixed point theorem on complete metric spaces and generalized metric spaces depend an another function . in this paper we obtained a fixed point theorem for extenended B-K contraction mapping on CGM-Spaces(Complete Generalized Metric) spaces depend on an another function.

1.1. Preliminaries

For our main results we need some of the following definitions.

Definition 1.1 (2)**.** Let (X, ρ) be a metric space. A mapping $A : X \longrightarrow X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ay_n\}$ is convergence then $\{y_n\}$ is also convergence. A is said sub sequentially convergent if we have , for every sequence $\{y_n\}$, if $\{Ay_n\}$ is convergence then $\{y_n\}$ has a convergent sub sequence.

Definition 1.2 (1)**.** Let X be a non empty set .Suppose that the map $\rho : X \times X \longrightarrow N$ satisfies the following:

(1) $0 \leq \rho(\alpha, \beta)$ for all $\alpha, \beta \in X$ and $\rho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;

(2) $\rho(\alpha, \beta) = \rho(\beta, \alpha)$ for all $\alpha, \beta \in X$;

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(3) $\rho(\alpha,\beta) \le \rho(\alpha,\xi)+\rho(\xi,\gamma)+\rho(\gamma,\beta)$ for all $\alpha,\beta \in X$ and for all distinct points $\gamma,\xi \in X \setminus \{\alpha,\beta\}$.
Then ρ is called a GM-(Generalized Metric) and (X,ρ) is a GM-Space (Generalized Metric Space).

2. Main Results

Now we obtain our main theorem.

Theorem 2.1. *Let (X,ρ) be a CGM-Space(Complete Generalized Metric Space) and $A,B : X \longrightarrow X$ be a mappings such that A is continuous one- to-one and subsequently convergent. If $a_1+a_2<1$ and*

$$\rho(ABx,ABy) \le a_1\rho(Ax,Ay)+a_2\rho(Ax,ABx)+\rho(Ay,ABy), \tag{1}$$

for all $x,y \in X$, then B has a unique fixed point. Also if A is sequentially convergent, then for every $x_0 \in X$ the sequence of iterates $\{B^n x_0\}$ converges to this fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point in X. We define the iterative sequence $\{x_n\}$ by

$$\begin{aligned} x_{n+1} = Bx_n =& B^n x_0, n=1,2,3\dots Usingby(1)wegetthat \\ \rho(Ax_n,Ax_{n+1}) =& \rho(ABx_{n-1},ABx_n), \\ \le& a_1\rho(Ax_{n-1},Ax_n)+a_2\rho(Ax_{n-1},ABx_{n-1})+\rho(Ax_n,ABx_n), \\ =& a_1\rho(Ax_{n-1},Ax_n)+a_2\rho(Ax_{n-1},Ax_n)+\rho(Ax_n,Ax_{n+1}), \\ =& (a_1+a_2)\rho(Ax_{n-1},Ax_n)+a_2\rho(Ax_n,Ax_{n+1}, \\ \le& (a_1+a_2)/(1-a_2)\rho(Ax_{n-1},Ax_n), \end{aligned}$$

$$\le h(\rho(Ax_{n-1},Ax_n). \tag{2}$$

Where , $h=a_1+a_2)/(1-a_2)<1$.
By the similar argument we get that

$$\rho(Ax_n,Ax_{n+1}) \le h\rho(Ax_{n-1},Ax_n) \le h^2\rho(Ax_{n-2},Ax_{n-1}) \le ... \le h^n\rho(Ax_{n-1},Ax_n). \tag{3}$$

By (3) fro all $m,n \in N$ such that $m>n$, we get that

$$\begin{aligned} \rho(Ax_m,Ax_n) \le& \rho(Ax_m,Ax_{m-1})+\rho(Ax_{m-1},Ax_{m-2})+...+\rho(Ax_{n+1},Ax_n), \\ \le& h^{m-1}+h^{m-2}+...+h^n\rho(Ax_0,Ax_1), \end{aligned}$$

$$\le h^n/(1-h)\rho(Ax_0,Ax_1). \tag{4}$$

Letting $m,n \longrightarrow \infty$ in (4) we get that $\{Ax_n\}$ is a Cauchy sequence and since X is complete there exists $p \in X$ such that

$$lim_{n\to\infty}Ax_n = p. \tag{5}$$

Since A is a sub sequentially convergent $\{x_n\}$ has a convergent sub sequence.So there exists $q \in X$ and $\{x_{n(r)}\}$, $r=1,2,,,\infty$ such that $lim_{r\to\infty}Ax_{n(r)}=q$.

$$Since\ A\ is\ continuous\ and\ lim_{r\to\infty}Ax_{n(r)} = Aq. \tag{6}$$

By (5) and (6) we conclude that $Aq = p$.

$$\rho(ABq, Aq) \leq \rho(ABq, AB^{n(r)}x_0) + \rho(AB^{n(r)}x_0, AB^{n(r)+1}x_0) + \rho(AB^{n(r)+1}x_0, Aq),$$

$$\leq a_1\rho(Aq, AB^{n(r)-1}x_0) + a_2[\rho(Aq, ABq) + \rho(AB^{n(r)-1}x_0, AB^{n(r)}x_0)] + h^{n(r)}\rho(Ax_1, Ax_0) + \rho(Ax_{n(r)+1}, Aq),$$

$$\leq a_1\rho(Aq, AB^{n(r)-1}x_0) + a_2\rho(Aq, ABq) + a_2h^{n(r)}\rho(Ax_0, Ax_1) + h^{n(r)}\rho(Ax_1, Ax_0) + \rho(Ax_n(r) + 1, Aq),$$

$$(1 - a_2)\rho(Aq, ABq) \leq a_1\rho(Aq, AB^{n(r)-1}x_0) + (a_2 + 1)h^{n(r)}\rho(Ax_0, Ax_1) + \rho(Ax_n(r) + 1, Aq),$$

$$\rho(Aq, ABq) \leq a_1/(1 - a_2)\rho(Aq, AB^{n(r)-1}x_0) + (a_2 + 1)/(1 - a_2)h^{n(r)}\rho(Ax_0, Ax_1)$$

$$+ 1/(1 - a_2)\rho(Ax_n(r) + 1, Aq). \longrightarrow 0 as r \longrightarrow \infty.$$

Thus, $\rho(Aq, ABq) = 0$, that is $ABq = Aq$. Since A is one- to- one, that implies, $Bq = q$. Therefore B has a fixed point. Since (1) holds and A is one –to- one. Therefore B has a unique fixed point.

Now if A is sequentially convergent, by replacing $\{n\}$ by $\{n(r)\}$ we conclude that $\lim_{r\to\infty} x_n = q$ and this shows that $\{x_n\}$ converges to the fixed point of B.

□

Remark 2.1. *If we take $a_1 = 0$ and $a_2 = \lambda$ in the above theorem 2.1 we can get the theorem 2.1 of [5].*

3. Conclusion

In this paper our results are generalized results and are more general than the results of [5].

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