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Fixed Point Results for General Integral Type Contraction Mappings in Cone Metric Spaces with Banach Algebra

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Abstract: The aim of this paper is to present a novel concept of generalized integral type contraction mapping in relation to a cone. The approach developed by F. Khojasteh is ultimately expanded under specific new contractive conditions of integral mapping to demonstrate fixed point results within the framework of cone metric spaces.

Key words: Banach algebra, generalized integral type contraction mapping, cone metric space.

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1. Introduction

Huang and Zhang [4] introduced the cone metric space **(CMS)** in 2007. They proved several fixed point theorems **(FPT)** in this space by replacing the real numbers with an ordered Banach space **(BS)**. Many **(FPT)** have been proven by numerous writers who have studied this topic (see [1], [5], [11], [3]). Branciari [2] first expanded Banach **(FPT)** and introduced the contractive condition of integral type. Subsequently, in **(CMS)**, Khojasteh and et.al. [6] presented the concept of a cone integrable function and provided a proof of Branciari's theorem.

This paper's goal is to apply the idea of Khojasteh [6] to a few novel integral-type contractive conditions in **(CMS)**.

1.1. Preliminaries

We can prove the main results with the help of the following definitions and lemmas of **(CMS)** and **(BA)**.

Definition 1.1. (See[12][10]) Consider $\mathcal{A}$ is always a real **(BA)**, which means that $\mathcal{A}$ is a **(BS)** whereby a multiplication operation has defined and applied the subsequent characteristics for every $\zeta, \xi, \upsilon \in \mathcal{A}$ and α element of $\mathbb{R}$.

(1) $\zeta(\xi\upsilon) = (\zeta\xi)\upsilon$;

(2) $\zeta\xi + \zeta\upsilon = \zeta(\xi + \upsilon)$ and $\zeta\upsilon + \xi\upsilon = (\zeta + \xi)\upsilon$;

(3) $(\alpha\zeta)\xi = \alpha(\zeta\xi) = \zeta(\alpha\xi)$;

(4) $||\zeta\xi|| \leq ||\zeta||.||\xi||$.

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In the framework of a (BA), we postulate the presence of a unit (or multiplicative identity) denoted as e, satisfying the condition that $e\zeta = \zeta e = \zeta$ for every element $\zeta \in A$. An element ζ belonging to A is deemed invertible if there exists an element $\xi \in A$ that serves as its inverse, fulfilling the equation $\zeta\xi = \xi\zeta = e$. The inverse of ζ is represented as ζ^{-1}.

Proposition 1.1. *(See[12],[10]) Let $\mathcal{A}$ denote a **(BA)** equipped within a unit element e, and let ζ be an element of $\mathcal{A}$. If the spectral radius $\rho(\zeta)$ of the element $\zeta < 1$, that is,*

$$\rho(\zeta) = \lim_{n\to\infty} ||\zeta_n||^{\frac{1}{n}} = \inf ||\zeta^n||^{\frac{1}{n}} < 1$$

Consequently, the expression $(e-\zeta)$ is invertible, and its inverse is given by $(e-\zeta)^{-1} = \sum_{i=0}^{\infty} \zeta^i$.

A subset $\mathcal{P}$ of $\mathcal{A}$ is referred to as a cone if

(1) $\mathcal{P}$ is non-empty closed and $\{\theta, e\} \subset \mathcal{P}$, where θ is $\mathcal{A}$'s zero vector;

(2) $\mathcal{P}\mathcal{P} = \mathcal{P}^2 \subset \mathcal{P}$;

(3) For any non-negative real numbers β and α exists such that $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$,

(4) $(-\mathcal{P}) \cap (\mathcal{P}) = \{\theta\}$.

For a specified cone $\mathcal{P}$ subset of $\mathcal{A}$, a partial ordering $\preceq$ can be established in relation to $\mathcal{P}$ such that $\zeta \preceq \xi$ holds iff $\xi - \zeta \in \mathcal{P}$. The symbol $\zeta \ll \xi$ is used to indicate that $\xi - \zeta \in \mathcal{P}^o$, where $\mathcal{P}^o$ represents the interior of the cone $\mathcal{P}$.

The cone $\mathcal{P}$ is referred to as usually if there is a constant $\mathcal{K} > 0$ so that for any $\alpha, \beta \in \mathcal{A}$, the condition $\alpha \preceq \beta$ leads to the conclusion that $||\alpha|| \leq \mathcal{K}||\beta||$.

The smallest positive value of $\mathcal{K}$ that satisfies the aforementioned inequality is referred to as the normal constant (refer to [4]). It is important to note that for any normal cone P, the condition $\mathcal{K} \geq 1$ holds (see [11]). In the subsequent discussion, we will assume that $\mathcal{P}$ represents a cone within a real **(BA)**$\mathcal{A}$, where $\mathcal{P}^o \neq \phi$ (indicating that the cone $\mathcal{P}$ is solid) and that $\preceq$ denotes the partial ordering associated with $\mathcal{P}$.

Definition 1.2. (See[7][4][8]) Let $\mathcal{U}$ represent a non-empty set. Assume that a function $d_c : \mathcal{U} \times \mathcal{U} \to \mathcal{A}$ fulfills the following conditions:

(1) For all $\zeta, \xi \in \mathcal{X}$, $\theta \preceq d_c(\zeta, \xi)$ and $d_c(\zeta, \xi) = \theta$ only in the event that $\zeta = \xi$;

(2) $d_c(\zeta, \xi) = d_c(\xi, \zeta), \forall \zeta, \xi \in \mathcal{U}$;

(3) $d_c(\zeta, \xi) \preceq d_c(\zeta, \upsilon) + d_c(\upsilon, \xi)$ for each $\zeta, \xi, \upsilon \in \mathcal{X}$.

A **(CMS)** on the set $\mathcal{U}$ is denoted by d, and the pair $(\mathcal{U}, d_c)$ is referred to as a **(CMS)** over the **(BA)** $\mathcal{A}$ (abbreviated as **CMSBA**). It is important to observe that for every pair of elements $\zeta, \xi \in \mathcal{U}$, the value $d_c(\zeta, \xi)$ belongs to the set $\mathcal{P}$.

Definition 1.3. (See[4],[9]) Let $(\mathcal{U}, d_c)$ represent a **(CMS)**, where $\zeta \in \mathcal{U}$ and $\{\zeta_n\}$ denotes a sequence within $\mathcal{U}$. Consequently:

(1) The sequence $\{\zeta_n\}$ is said to converge to ζ if, for every $c \in \mathcal{A}$ with $\theta \ll c$, there exists an integer $n_0 \in \mathbb{N}$ so that $c \gg d_c(\zeta_n, \zeta)$ holds for every $n_0 < n$. That is expressed as $\lim_{n\to\infty} \zeta_n = \zeta$ or $\zeta_n \to \zeta$ as $n \to \infty$.

(2) The sequence $\{\zeta_n\}$ is classified as a Cauchy sequence if, for every $c \in \mathcal{A}$ where $c \gg \theta$, there exists an integer $n_0 \in \mathbb{N}$ so that $d_c(\zeta_n, \zeta_m) \ll c$ holds true for all $n_0 < n, m$.

(3) A complete **(CMS)** is defined as $(\mathcal{U}, d_c)$ if every Cauchy sequence contained in $\mathcal{U}$ converges..

Lemma 1.1. *(See[4]) Let $(\mathcal{U}, d_c)$ represent a **(CMS)**, and let $\mathcal{P}$ denote a normal cone characterized by a usual constant $\mathcal{K}$. Consider the sequence ζ_n within the space $\mathcal{U}$. Then.*

(1) The sequence ζ_n is said to converge to ζ iff the distance $d_c(\zeta_n, \zeta)$ approaches 0 as n approaches infinity.

(2) A sequence ζ_n is classified as a Cauchy sequence iff the distance $d_c(\zeta_n, \zeta_m)$ approaches zero as both m and n tend to infinity.

Definition 1.4. (See[4],[9]) Let $(\mathcal{U}, d_c)$ represent a **(CMS)**. If every Cauchy sequence within $\mathcal{U}$ converges, then $\mathcal{U}$ is referred to as a complete **(CMS)**.

Lemma 1.2. *(See[4])Let $(\mathcal{U}, d_c)$ represent a **(CMS)**, and let $\mathcal{P}$ denote a normal cone characterized by a normal constant $\mathcal{K}$. Consider the sequences ζ_n and ζ_n within the space $\mathcal{U}$.*

(1) If the sequence ζ_n approaches the value ζ and simultaneously converges to the value ξ, it follows that ζ must equal ξ. This indicates that the limit of the sequence ζ_n is unique, and it is evident that the limit of the sequence ξ_n is also unique.

(2) The sequences ζ_n and ξ_n converge to ζ and ξ, respectively as n approaches infinity, then the distance $d_c(\zeta_n, \zeta_m)$ converges to $d_c(\zeta, \xi)$ as n approaches infinity.

Example 1.1. *[4] Consider a **(BS)** $E = R^2$, the cone $\mathcal{P} = \{(\zeta, \xi) \in E | \zeta, \xi \geq 0\} \subset \mathbb{R}^2, \mathcal{U} = R$ and $d_c : \mathcal{U} \times \mathcal{U} \to E$ for $\beta \geq 0$ a constant such that $d_c(\zeta, \xi) = (|\zeta - \xi|, \alpha|\zeta - \xi|)$, then $(\mathcal{U}, d_c)$ is a **(CMS)**.*

The subsequent lemmas and findings will be instrumental in establishing the primary result.

Theorem 1.1. *[4]Let $(\mathcal{U}, d_c)$ be a complete cone metric space, and let $\mathcal{P}$ denote a normal cone characterized by a normal constant K. Suppose the mapping $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ satisfies the contractive condition*

$$d_c(\mathcal{J}\zeta, \mathcal{J}\xi) \leq \beta d_c(\zeta, \xi)$$

for all $\zeta, \xi \in \mathcal{U}$, where $\beta \in (0, 1)$. Consequently, $\mathcal{J}$ possesses a singular fixed point. $\zeta_0 \in \mathcal{U}$. For all $\zeta \in \mathcal{U}$, sequence $\{\mathcal{J}^n(\zeta)\}$ converges to ζ_0.

In 2002, Branciari [2] presented a comprehensive contractive condition of integral type within the framework of **(CMS)** as detailed below.

Theorem 1.2. *[2] Let $(\mathcal{U}, d_c)$ represent a complete metric space, where β is a value in the interval $(0, 1)$, and consider the mapping $\mathcal{J} : \mathcal{U} \to \mathcal{U}$, which holds for all elements $\zeta, \xi \in \mathcal{U}$,*

$$\int_0^{d_c(\mathcal{J}\zeta, \mathcal{J}\xi)} \phi(t)\, dt \leq \beta \int_0^{d_c(\zeta, \xi)} \phi(t)\, dt$$

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a nonnegative and Lebesgue-integrable function that is summable on every compact subset of $[0, +\infty)$. It is required that for every $\epsilon > 0$, the integral $\int_0^{\epsilon} \phi(t)\, dt$ is greater than zero. Under these conditions, the function f possesses a unique fixed point $\delta \in \mathcal{U}$, such that $\lim_{n\to\infty} \mathcal{J}^n \zeta = \delta$.

In this section, we introduce an innovative concept of an integral associated with a cone and present Branciari's findings as outlined in **(CMS)** regarding Banach algebra.

Definition 1.5. [6]Let us consider that $\mathcal{P}$ represents a normal cone in the space E. Let $\delta, \eta \in E$ and $\delta < \eta$. We define

$$[\delta, \eta = \{\zeta \in E : \zeta = t\eta + (1-t)\delta, t \in [0,1]\},$$

$$[\delta, \eta) = \{\zeta \in E : \zeta = t\eta + (1-t)\delta, t \in [0,1)\},$$

Definition 1.6. [6] The set $\mathcal{P}_1 = \{\delta = \zeta_0\, \zeta_1\, \zeta_2 .... \zeta_n = \eta\}$ is designated a divisin of $[\delta, \eta]$ iff the sets $\{[\zeta_{j-1}, \zeta_j]\}_{j=1}^{n}$ the sets are mutually exclusive and $[\delta, \eta] = \{\bigcup_{j-1}^{n} [\zeta_{j=1}, \zeta_j)\} \cup \{\eta\}$.

Definition 1.7. [6]Let $\mathcal{P}_1 = \{\delta = \zeta_0\zeta_1\zeta_2....\zeta_n = \eta\}$ be a partition of $[\delta, \eta]$ and $\phi = [\delta, \eta] \to \mathcal{P}$ an increasing function is defined as one that consistently rises. We establish the concepts of cone lower sum and cone upper sum as follows.

$$L_n^{con}(\phi, \mathcal{P}_1) = \sum_{j=0}^{n-1} \phi(\zeta_j) \|\zeta_j - \zeta_{j+1}\|$$

$$U_n^{con}(\phi, \mathcal{P}_1) = \sum_{j=0}^{n-1} \phi(\zeta_{j+1}) \|\zeta_j - \zeta_{j+1}\|$$

respectively.

The function ϕ is referred to as a cone integrable function on the interval $[\delta, \eta]$ if and only if it holds true for every partition $\mathcal{P}_1$ of the interval $[\delta, \eta]$.

$$\lim_{n\to\infty} L_n^{con}(\phi, \mathcal{P}_1) = S^{con} = \lim_{n\to\infty} U_n^{con}(\phi, \mathcal{P}_1)$$

where S^{con} is unique. We shall write ${}^{con} = \int_\delta^\eta \phi d\mathcal{P}$ or $\int_\delta^\eta \phi(t) d\mathcal{P}(t)$.

Lemma 1.3. *[6]If $[\delta, \eta] \subseteq [\delta, \gamma]$ then $\int_\delta^\eta \phi\, d\mathcal{P} \le \int_\delta^\gamma \phi\, dp$ for $\phi \in l^1(\mathcal{U}, \mathcal{P})$*

$$\int_\delta^\eta (h\phi_1 + g\phi_2)\, d\mathcal{P} = h \int_\delta^\eta \phi_1 d\mathcal{P} + g \int_\delta^\eta \phi_2 d\mathcal{P}$$

for $\phi_1, \phi_2 \in l^1(\mathcal{U}, \mathcal{P})$ and $g, h \in R$. Where $l^1(\mathcal{U}, \mathcal{P})$ the notation represents the collection of all functions that are integrable with respect to the cone.

Definition 1.8. [6] A function $\phi : \mathcal{P} \to E$ is classified as a subadditive cone integrable function iff for all $\delta, \eta \in \mathcal{P}$

$$\int_0^{\delta+\eta} \phi\, d\mathcal{P} \le \int_0^\delta \phi\, d\mathcal{P} + \int_0^\eta \phi\, d\mathcal{P}$$

2. Main Results

Theorem 2.1. *Let $(\mathcal{U}, d_C)$ represent a complete **(CMS)** equipped with a normal cone denoted as $\mathcal{P}$. Consider the function $\phi : \mathcal{P} \to \mathcal{P}$, which is a nonvanishing and subadditive cone integrable mapping defined on every interval $[\delta, \eta] \subset \mathcal{P}$ and $\int_0^\epsilon \phi\, d\mathcal{P} \gg 0, \epsilon \gg 0$. Let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ be a mapping such that*

$$\int_0^{d_c(\mathcal{J}\delta,\ \mathcal{J}\eta)} \phi\, d\mathcal{P} \preceq \mu_1 \int_0^{d_c(\delta,\ \mathcal{J}\delta)+d_c(\eta,\ \mathcal{J}\eta)} \phi\, d\mathcal{P} + \mu_2 \int_0^{d_c(\delta,\ \mathcal{J}\eta)+d_c(\mathcal{J}\delta,\ \eta)} \phi\, d\mathcal{P}$$

for each $\delta, \eta \in \mathcal{U}$ *and* $(\mu_1 + \mu_2) < \frac{1}{2}.\mu_1, \mu_2 \in (0, \frac{1}{2})$. *Then* $\mathcal{J}$ *it possesses a distinct fixed point in* $\mathcal{U}$.

Proof. Suppose $\delta \in \mathcal{U}$ and $\delta_1 \in \mathcal{U}$ in a manner that $\delta_1 = \mathcal{J}(\delta)$. Let $\delta_2 \in \mathcal{U}$ such that $\delta_2 = \mathcal{J}(\delta)$. Continuing in this way we can define $\delta_{n+1} = \mathcal{J}(\delta_n) = \mathcal{J}^n(\delta)$

$$\begin{aligned}
\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} &= \int_0^{d_c(\mathcal{J}\delta_n,\ \mathcal{J}\delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq \mu_1 \int_0^{d_c(\delta_n,\ \mathcal{J}\delta_n)+d_c(\delta_{n-1},\ \mathcal{J}\delta_{n-1})} \phi\, d\mathcal{P} + \mu_2 \int_0^{d_c(\delta_n,\ \mathcal{J}\delta_{n-1})+d_c(\mathcal{J}\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq \mu_1 \int_0^{d_c(\delta_n,\ \delta_{n+1})+d_c(\delta_{n-1},\ \delta_n)} \phi\, d\mathcal{P} + \mu_2 \int_0^{d_c(\delta_n,\ \delta_n)+d_c(\delta_{n+1},\ \delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq \mu_1 \int_0^{d_c(\delta_n,\ \delta_{n+1})+d_c(\delta_{n-1},\ \delta_n)} \phi\, d\mathcal{P} + \mu_2 \int_0^{d_c(\delta_{n+1},\ \delta_n)+d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq (\mu_1 + \mu_2) \left[\int_0^{d_c(\delta_n,\ \delta_{n+1})+d_c(\delta_{n-1},\ \delta_n)} \phi\, d\mathcal{P} + \int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} \right] \\
&\preceq \mu \int_0^{d_c(\delta_{n-1},\ \delta_n)+d_c(\delta_n,\ \delta_{n+1})} \phi\, d\mathcal{P}
\end{aligned}$$

Where $\mu = \frac{\mu_1+\mu_2}{1-(\mu_1+\mu_2)}$ .Since ϕ is cone subadditive, so

$$\begin{aligned}
\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} &\preceq \mu \int_0^{d_c(\delta_{n-1},\ \delta_n)} \phi\, d\mathcal{P} + \mu \int_0^{d_c(\delta_n,\ \delta_{n+1})} \phi\, d\mathcal{P} \\
&\preceq \mu(e-\mu)^{-1} \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq \beta \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P}
\end{aligned}$$

Where $\beta = \mu(e-\mu)^{-1}$

$$\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} \preceq ....\beta^n \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P}$$

Now

$$\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} \preceq \beta^n \int_0^{d_c(\mathcal{J}(\delta),\ \delta)} \phi\, d\mathcal{P}$$

Since $0 \le \beta < 1$ and $\int_0^{\epsilon} \phi\, d\mathcal{P} \gg 0, \epsilon \gg 0$, so

$$\lim_{n\to\infty} \int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} = \theta$$

which implies, that $\lim_{n\to\infty} d_c(\delta_{n+1},\ \delta_n) = \theta$

To demonstrate that the sequence $\{\delta_n\}$ is a Cauchy sequence, we will establish that.

$$\lim_{m,n\to\infty} d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n)) = \theta$$

By triangle inequality

$$\int_0^{d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n))} \phi\, d\mathcal{P} \preceq \int_0^{d_c(\mathcal{J}(\delta_n),\ \mathcal{J}(\delta_{n+1}))+d_c(\mathcal{J}(\delta_{n+1}),\ \mathcal{J}(\delta_{n+2}))+.....d_c(\mathcal{J}(\delta_{m-1}),\ \mathcal{J}(\delta_m))} \phi\, d\mathcal{P}$$

Consequently, by the principle of sub-additivity of ϕ, we obtain.

$$\begin{aligned}\int_0^{d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n))} \phi\, d\mathcal{P} &\preceq \int_0^{d_c(\mathcal{J}(\delta_n),\ \mathcal{J}(\delta_{n+1}))} \phi\, d\mathcal{P} + ... + \int_0^{d_c(\mathcal{J}(\delta_{m-1}),\ \mathcal{J}(\delta_m))} \phi\, d\mathcal{P} \\ &\preceq (\beta^n + \beta^{n-1} + ... + \beta^m) \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P} \\ &\preceq \beta^n (e-\beta)^{-1} \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P} \to 0 (n\to\infty)\end{aligned}$$

Thus

$$\lim_{m,n\to\infty} d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n)) = \theta$$

This indicates that the sequence $\{\delta_n\}$ is a Cauchy sequence. Given that $\mathcal{U}$ is a complete **(CMS)**, it follows that the sequence $\{\delta_n\}$ converges to some limit $\delta_0 \in \mathcal{U}$. Ultimately, since.

$$\int_0^{d_c(\delta_{n+1},\ \mathcal{J}(\delta_0))} \phi\, d\mathcal{P} = \int_0^{d_c(\mathcal{J}(\delta_n),\ \mathcal{J}(\delta_0))} \phi\, d\mathcal{P} \preceq \beta \int_0^{d_c(\delta_n,\ \delta_0)} \phi\, d\mathcal{P}$$

Thus $\lim_{m,n\to\infty} d_c((\delta_{n+1}),\ (\delta_0)) = \theta$. This means that $\mathcal{J}(\delta_0) = \delta_0$. If δ_0, η_0 are two separate fixed points of $\mathcal{J}$, then

$$\int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P} = \int_0^{d_c(\mathcal{J}(\delta_0),\ \mathcal{J}(\eta_0))} \phi\, d\mathcal{P} \preceq \beta \int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P}$$

This presents a contradiction. Therefore, $\mathcal{J}$ possesses a unique fixed point, denoted as $\delta_0 \in \mathcal{U}$. □

Theorem 2.2. *Let $(\mathcal{U}, d_C)$ represent a complete **(CMS)** equipped with a normal cone denoted as $\mathcal{P}$. Consider the function $\phi : \mathcal{P} \to \mathcal{P}$, which is a nonvanishing and subadditive cone integrable mapping defined on every interval $[\delta, \eta] \subset \mathcal{P}$ and $\int_0^{\epsilon} \phi\, d\mathcal{P} \gg 0, \epsilon \gg 0$. Let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ be a mapping such that*

$$\int_0^{d_c(\mathcal{J}\delta,\ \mathcal{J}\eta)} \phi\, d\mathcal{P} \preceq \mu \int_0^{\max\{d_c(\delta,\ \eta), d_c(\delta,\ \mathcal{J}\delta), d_c(\eta,\ \mathcal{J}\eta)\}} \phi\, d\mathcal{P}$$

for each $\delta, \eta \in \mathcal{U}$ and $\mu \in [0, 1)$. Then $\mathcal{J}$ possesses a distinct fixed point in $\mathcal{U}$.

Proof. Suppose $\delta \in \mathcal{U}$ and $\delta_1 \in \mathcal{U}$ such that $\delta_1 = \mathcal{J}(\delta)$. Let $\delta_2 \in \mathcal{U}$ such that $\delta_2 = \mathcal{J}(\delta)$. Continuing in this way we can define $\delta_{n+1} = \mathcal{J}(\delta_n) = \mathcal{J}^n(\delta)$

$$\begin{aligned}
\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} &= \int_0^{d_c(\mathcal{J}\delta_n,\ \mathcal{J}\delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq \mu \int_0^{\max\{d_c(\delta_n,\ \delta_{n-1}), d_c(\delta_n,\ \mathcal{J}\delta_n), d_c(\delta_{n-1},\ \mathcal{J}\delta_n\}} \phi\, d\mathcal{P} \\
&\preceq \mu \int_0^{\max\{d_c(\delta_n,\ \delta_{n-1}), d_c(\delta_n,\ \delta_{n+1}), d_c(\delta_{n-1},\ \delta_n\}} \phi\, d\mathcal{P} \\
&\preceq \mu \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\
&\preceq \mu^n \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P}
\end{aligned}$$

Since $0 \leq \mu < 1$ and $\int_0^{\epsilon} \phi\, d\mathcal{P} \gg 0, \epsilon \gg 0$, so

$$\lim_{n\to\infty} \int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} = \theta$$

which implies, that $\lim_{n\to\infty} d_c(\delta_{n+1},\ \delta_n) = \theta$

To demonstrate that the sequence $\{\delta_n\}$ is a Cauchy sequence, we will establish that.

$$\lim_{m,n\to\infty} d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n)) = \theta$$

By triangle inequality

$$\int_0^{d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n))} \phi\, d\mathcal{P} \preceq \int_0^{d_c(\mathcal{J}(\delta_n),\ \mathcal{J}(\delta_{n+1})) + d_c(\mathcal{J}(\delta_{n+1}),\ \mathcal{J}(\delta_{n+2})) + \ldots.. d_c(\mathcal{J}(\delta_{m-1}),\ \mathcal{J}(\delta_m))} \phi\, d\mathcal{P}$$

By virtue of the sub-additivity property of ϕ, we obtain.

$$\begin{aligned}
\int_0^{d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n))} \phi\, d\mathcal{P} &\preceq \int_0^{d_c(\mathcal{J}(\delta_n),\ \mathcal{J}(\delta_{n+1}))} \phi\, d\mathcal{P} + \ldots. + \int_0^{d_c(\mathcal{J}(\delta_{m-1}),\ \mathcal{J}(\delta_m))} \phi\, d\mathcal{P} \\
&\preceq (\mu^n + \mu^{n-1} + \ldots.. + \mu^m) \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P} \\
&\preceq \mu^n (e - \mu)^{-1} \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P} \to 0 (n \to \infty)
\end{aligned}$$

Thus

$$\lim_{m,n\to\infty} d_c(\mathcal{J}(\delta_m),\ \mathcal{J}(\delta_n)) = \theta$$

This indicates that the sequence $\{\delta_n\}$ is a Cauchy sequence. Given that $\mathcal{U}$ is a complete **(CMS)**, it follows that the sequence $\{\delta_n\}$ converges to some limit $\delta_0 \in \mathcal{U}$. Ultimately, since.

$$\int_0^{d_c(\delta_{n+1},\ \mathcal{J}(\delta_0))} \phi\, d\mathcal{P} = \int_0^{d_c(\mathcal{J}(\delta_n),\ \mathcal{J}(\delta_0))} \phi\, d\mathcal{P} \preceq \beta \int_0^{d_c(\delta_n,\ \delta_0)} \phi\, d\mathcal{P}$$

Thus $\lim_{m,n\to\infty} d_c((\delta_{n+1}),\ (\delta_0)) = \theta$. This means that $\mathcal{J}(\delta_0) = \delta_0$. If δ_0, η_0 are two separate fixed points of $\mathcal{J}$, then

$$\int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P} = \int_0^{d_c(\mathcal{J}(\delta_0),\ \mathcal{J}(\eta_0))} \phi\, d\mathcal{P} \preceq \beta \int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P}$$

This presents a contradiction. Therefore, $\mathcal{J}$ possesses a unique fixed point, denoted as $\delta_0 \in \mathcal{U}$. □

Theorem 2.3. *Let $(\mathcal{U}, d_C)$ represent a complete* ***(CMS)*** *equipped with a normal cone denoted as $\mathcal{P}$. Consider the function $\phi : \mathcal{P} \to \mathcal{P}$, which is a nonvanishing and subadditive cone integrable mapping defined on every interval $[\delta, \eta] \subset \mathcal{P}$ and $\int_0^{\epsilon} \phi\, d\mathcal{P} \gg 0, \epsilon \gg 0$. Let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ be a mapping such that*

$$\int_0^{d_c(\mathcal{J}\delta,\ \mathcal{J}\eta)} \phi\, d\mathcal{P} \preceq \mu \int_0^{d_c(\delta,\ \eta)+d_c(\delta,\ \mathcal{J}(\delta))+d_c(\eta,\ \mathcal{J}(\eta))} \phi\, d\mathcal{P}$$

for each $\delta, \eta \in \mathcal{U}$ and $\mu \in (0, \frac{1}{3})$. Then $\mathcal{J}$ demonstrates that $\mathcal{J}$ possesses a singular fixed point in $\mathcal{U}$.

Proof. Assume $\delta \in \mathcal{U}$ and $\delta_1 \in \mathcal{U}$ such that $\delta_1 = \mathcal{J}(\delta)$. Let $\delta_2 \in \mathcal{U}$ such that $\delta_2 = \mathcal{J}(\delta)$. Continuing in this way we can define $\delta_{n+1} = \mathcal{J}(\delta_n) = \mathcal{J}^n(\delta)$

$$\begin{aligned}\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} &= \int_0^{d_c(\mathcal{J}\delta_n,\ \mathcal{J}\delta_{n-1})} \phi\, d\mathcal{P} \\ &\preceq \mu \int_0^{d_c(\delta_n,\ \delta_n)+d_c(\delta_{n-1},\ \delta_{n+1})+d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\ &\preceq \mu \int_0^{d_c(\delta_{n-1},\ \delta_{n+1})} \phi\, d\mathcal{P} + \mu \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P}\end{aligned}$$

Using triangular inequality and cone subadditivity.

$$\begin{aligned}\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} &\preceq \mu \int_0^{d_c(\delta_{n-1},\ \delta_n)} \phi\, d\mathcal{P} + \mu \int_0^{d_c(\delta_n,\ \delta_{n+1})} \phi\, d\mathcal{P} + \mu \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\ &\preceq 2\mu(e-\mu)^{-1} \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\ &\preceq \beta \int_0^{d_c(\delta_n,\ \delta_{n-1})} \phi\, d\mathcal{P} \\ &\preceq ......\beta^n \int_0^{d_c(\delta_1,\ \delta_0)} \phi\, d\mathcal{P} \\ &= \beta^n \int_0^{d_c(\mathcal{J}(\delta),\ \delta)} \phi\, d\mathcal{P}\end{aligned}$$

Where $\beta = 2\mu(e-\mu)^{-1}$.If $0 < 2\mu(e-\mu)^{-1} < 1$ that is $\mu < \frac{1}{3}$

$$\int_0^{d_c(\delta_{n+1},\ \delta_n)} \phi\, d\mathcal{P} = \theta$$

which implies that

$$\lim_{n\to\infty} d_c(\delta_{n+1},\ \delta_n) = \theta$$

It can be readily demonstrated, similar to theorems (2.1), that the sequence $\{\delta_n\}$ qualifies as a Cauchy sequence. Furthermore, the completeness of the space **CMS** $\mathcal{U}$ guarantees the existence of an element $\delta_0 \in \mathcal{U}$ for which the limit $\lim_{n\to\infty} \delta_n = \delta_0$ holds true.Now

$$\begin{aligned}
\int_0^{d_c(\mathcal{J}(\delta_0),\ \delta_{n+1})} \phi\, d\mathcal{P} &= \int_0^{d_c(\mathcal{J}(\delta_0),\ \mathcal{J}(\delta_n))} \phi\, d\mathcal{P} \\
&\preceq \mu \int_0^{d_c(\delta_0,\ \delta_{n+1})+d_c(\delta_n,\ \mathcal{J}(\delta_0))+d_c(\delta_0,\ \delta_n)} \phi\, d\mathcal{P} \\
&\preceq \mu \int_0^{d_c(\delta_0,\ \delta_{n+1})} \phi\, d\mathcal{P} + \mu \int_0^{d_c(\delta_n,\ \mathcal{J}(\delta_0))} \phi\, d\mathcal{P} + \mu \int_0^{d_c(\delta_0,\ \delta_n)} \phi\, d\mathcal{P}
\end{aligned}$$

As $n \to \infty$

$$\int_0^{d_c(\mathcal{J}(\delta_0),\ \delta_0)} \phi\, d\mathcal{P} \preceq \mu \int_0^{d_c(\delta_0,\ \mathcal{J}(\delta_0))} \phi\, d\mathcal{P}$$

Given that $0 < \mu < \frac{1}{3}$, it follows that $\int_0^{d_c(\mathcal{J}(\delta_0),\ \delta_0)} \phi\, d\mathcal{P} = \theta$. This indicates that $d_c(\mathcal{J}(\delta_0), \delta_0) = \theta$, which further leads to the conclusion that $\mathcal{J}(\delta_0) = \delta_0$.

Let $\mathcal{J}$ have two fixed points δ_0 and η_0 i.e. $\mathcal{J}(\delta_0) = \delta_0$ and $\mathcal{J}(\eta_0) = \eta_0$

$$\begin{aligned}
\int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P} &= \int_0^{d_c(\mathcal{J}(\delta_0),\ \mathcal{J}(\eta_0))} \phi\, d\mathcal{P} \\
&\preceq \mu \int_0^{d_c(\delta_0,\ \mathcal{J}(\eta_0))+d_c(\eta_0,\ \mathcal{J}(\delta_0))+d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P} \\
&\preceq 3\mu \int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P} \\
&= \theta
\end{aligned}$$

Since $0 < \mu < \frac{1}{3}$ therefore

$$\int_0^{d_c(\delta_0,\ \eta_0)} \phi\, d\mathcal{P} = \theta$$

This implies $d_c(\delta_0,\ \eta_0) = \theta$

$$\delta_0 = \eta_0$$

It demonstrates that $\mathcal{J}$ possesses a singular fixed point. □

Example 2.1. *Let $\mathcal{U} = [0,1]$ and let d_c denote the standard metric with $d_c(\delta,\eta) = \|\delta - \eta\|$. Clearly $(\mathcal{U}, d_c)$ is a complete* ***(CMS)****. Let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ be provided by $\mathcal{J}\delta = \frac{\delta}{2}$ for all $\delta \in [0,1]$. Once more, allow $\phi : R^+ \to R^+$ be provided by $\phi(t) = \frac{t^2}{2}$ for all $t \in R^+$. Then for each $\epsilon > 0$*

$$\int_0^{\epsilon} \phi(t)\, dt = \int_0^{\epsilon} \frac{t^2}{2}\, dt = \frac{\epsilon^3}{6} > 0$$

By setting $\mu = \frac{1}{16}$, *it can be readily confirmed that the condition of Theorem (2.1) is fulfilled, given that* $0 < \mu < 1$. *Consequently, this leads to the existence of a unique fixed point for* $\mathcal{J}$.

3. Application

In biological systems, like population models or ecosystems, interactions between different species can be intricate and multifaceted. The study of species stability and persistence in dynamic ecosystems, where growth rates and interactions are controlled by various factors, is aided by fixed point results in CMS. Integral type contractions are useful in modeling how populations stable in the face of varying environmental conditions throughout time.

4. Conclusion

In this study, we have formulated specific fixed point theorems for generalized integral type contraction mappings within the context of complete **(CMS)** over **(BA)**, utilizing Banach's principle. Additionally, we have explored the ramifications of our primary results. The findings articulated in this paper expand upon and enhance various elements of previous research documented in the literature.

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