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New Results on Behavior of Solutions of Certain System of Third-Order Non-linear Differential Equations

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Abstract: In this article, provisions are made for sufficient conditions that guaranteed uniform asymptotic stability of the trivial solution and uniform boundedness of all solutions to a class of third-order non-linear differential equations. The direct method of Lyapunov is used to establish our results. Numerical examples are given together with the graphical representation of their solutions by Maple software as a justification for our findings.

Key words: Third order, Stability, Boundedness, Lyapunov function.

1. Introduction

The main intention of this article is to employ the Lyapunov direct method to address the problem of stability and boundedness behavior of solutions to the following equation

$$X''' + r(t)\Psi(X,X')X'' + q(t)\Phi(X,X')X' + \eta H(X) = P(t,X,X',X''), \tag{1}$$

or its system form

$$\begin{aligned} X' &= Y,\\ Y' &= Z,\\ Z' &= -r(t)\Psi(X,Y)Z - q(t)\Phi(X,Y)Y - \eta H(X) + P(t,X,Y,Z), \end{aligned} \tag{2}$$

where $t \in \mathbb{R}^+ = [0,+\infty), \mathbb{R} = (-\infty,+\infty),\ X,Y,Z \in \mathbb{R}^n;\ H : \mathbb{R}^n \to \mathbb{R}^n$ is continuous in its argument displayed explicitly, and is such that, its $n \times n$ Jacobian matrix $J_h(X)$ as well as $n \times n$ matrices $\Psi(X,Y)$ and $\Phi(X,Y)$ are symmetric, positive definite and continuous in their respective arguments, $P : (\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n$; $r(t)$ and $q(t)$ are continuous functions of t, such that for constants $q_1, r_1,\ 1 \le q(t) \le q_1, 1 \le r(t) \le r_1$, $q'(t) \le 0, r'(t) \le 0,\ \eta$ is a positive constant and the prime($'$) indicate differentiation with respect to t. To ensure the existence and uniqueness of solutions of equation (1) or system (2), we assumed in addition to the continuity condition on Ψ, Φ, H and P, that they satisfied the Lipschitz condition with respect to their respective arguments, (see, Rao [21]). Equation (1) is a representation of a system of real third-order equation given by

$$\begin{aligned} &x_i''' + r(t)\textstyle\sum_{k=1}^n \psi_{ik}(x_1,x_2,\ldots,x_n,x_1',x_2',\ldots,x_n')x_k'' + q(t)\textstyle\sum_{k=i}^n \phi_{ik}(x_1,x_2,\ldots,x_n,x_1',x_2',\ldots,x_n')x_k' \\ &\quad + \eta h_i(x_1,x_2,\ldots,x_n) = p_i(t,x_1,x_2,\ldots,x_n,x_1',x_2',\ldots,x_n',x_1'',x_2'',\ldots,x_n''), \end{aligned} \tag{3}$$

$i = 1,2,3,\ldots,n$, and as usual, the functions $\phi_{ik}, \psi_{ik}, h_i, q, r$ and p_i are continuous.

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Before now, the study of qualitative behavior (stability, boundedness, convergence, periodicity and others) of solutions to differential equations has been receiving appreciable attention from many notable researchers for many decades, see the references. The reason for this may be attributed to the fact that, differential equations are of great applications in many areas of science and technology([9], [14], [22], [23]). In most of the works found in the literature on the qualitative behavior of solutions of differential equations, the direct method of Lyapunov was used. This method is highly effective and does not need the foreknowledge of the solutions of the differential equation being studied. The main idea of the method involves, constructing a positive definite function (called Lyapunov function), whose derivative with respect to an independent variable t along the solution path of the differential equation is negative semi-definite. The followings are some of the established results found in the literature based on the method.

Ultimate boundedness and existence of periodic solutions of the equation

$$X''' + AX'' + BX' + H(X) = P(t, X, X', X''),$$

were considered by Afuwape [6] and Meng [12], where A and B are $n \times n$ constant matrices. Later, Afuwape and Omeike [7], gave sufficient conditions for ultimate boundedness of solutions to

$$X''' + F(X'') + G(X') + H(X) = P(t, X, X', X'').$$

Tunç [25] provided some criteria for the stability and boundedness of solutions of

$$X''' + \Psi(X')X'' + BX' = P(t), \tag{4}$$

when $P(t) = 0$ and $P(t) \neq 0$ respectively. Omeike and Afuwape [16] in their work, proved certain result on the ultimate boundedness of the same equation (4). In 2014, Omeike [17] studied the global asymptotic stability of the trivial solution and boundedness of all solutions to

$$X''' + \Psi(X')X'' + \Phi(X)X' + cX = P(t),$$

where c is a positive constant. In a recent paper by Abdurasid *et al.* [2], conditions for ultimate boundedness of solutions to

$$X''' + \Psi(X')X'' + \Phi(X)X' + H(X) = P(t, X, X', X''),$$

were established with function $H(X)$ not necessarily differentiable. Qualitative properties of some other equations have also being studied by means of fixed point theorem(See, [1], [11]).

The motivation for this article comes from [2], [16], [17] and [25] where behavior of solutions of some third-order differential equations were examined as mentioned earlier. Our goal therefore, is to generalize and improve on their stability and boundedness results by considering a more general equation (1). It is worth noting that, no work has been done to the best of our knowledge in the literature regarding equation (1).

Remark 1.1.
If $n = 1$ in equation (1), we have the following scalar differential equation

$$x''' + r(t)\psi(x, x')x'' + q(t)\phi(x, x')x' + \eta h(x) = p(t, x, x', x''), \tag{5}$$

which to the best of our knowledge has not been considered in the literature. Equation (5) is a generalization of many scalar differential equations studied in the literature, see ([8], [13], [15], [18], [19], [20], [22], [26]) and other works cited in their references.

2. Preliminary Results

In this section, we state some algebraic results required in the proofs of our main results. The proofs of Lemmas (2.1) and (2.2) can easily be found in ([3]- [7], [10], [24]).

Lemma 2.1. *Let D be a real symmetric positive definite $n \times n$ matrix. Then for any X in $\mathbb{R}^n$, we have,*

$$\delta_d \|X\|^2 \leq \langle DX, X\rangle \leq \Delta_d \|X\|^2,$$

where δ_d and Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2. *Let $H(X)$ be a vector function for which $H(0) = 0$. $J_h(X)$ (the Jacobian matrix of $H(X)$), $\Psi(X,Y)$ and $\Phi(X,Y)$ appearing in (2) be continuous matrix functions and that the Jacobian matrices $J(\Phi(X,Y)Y|X)$ and $J(\Psi(X,Y)Y|X)$ of $\Phi(X,Y)$ and $\Psi(X,Y)$ are negative semi-definite. Then,*

(i) $\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \langle H(X), Y\rangle$,

(ii) $\frac{d}{dt}\int_0^1 \langle \sigma\Psi(X,\sigma Y)Y, Y\rangle d\sigma = \langle \Psi(X,Y)Y, Z\rangle + \int_0^1 \langle \sigma J(\Psi(X,\sigma Y)Y|X)Y, Y\rangle d\sigma,$

(iii) $\frac{d}{dt}\int_0^1 \langle \sigma\Phi(X,\sigma Y)Y, Y\rangle d\sigma = \langle \Phi(X,Y)Y, Z\rangle + \int_0^1 \langle \sigma J(\Phi(X,\sigma Y)Y|X)Y, Y\rangle d\sigma,$

(iv) $\int_0^1 \langle a_0 H(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \sigma\langle a_0 J_h(\sigma\tau X)X, X\rangle d\sigma d\tau.$

3. Stability Result

We state the following theorem for the case $P(t, X, Y, Z) \equiv 0$.

Theorem 3.1. *Suppose in addition to all the basic assumptions imposed on Ψ, $\Phi, J_h(X), r(t)$ and $q(t)$; $H(0) = 0$ and there exist some positive constants a_0, b_0, c_0, a_1, b_1 and c_1 such that the following conditions are satisfied for all X, $Y \in \mathbb{R}^n$:*

(i) $a_0 \leq \lambda_i(\Psi(X,Y)) \leq a_1$,

(ii) $b_0 \leq \lambda_i(\Phi(X,Y)) \leq b_1$,

(iii) $c_0 \leq \lambda_i(J_h(X)) \leq c_1$,

where $\lambda_i(\Psi(X,Y))$, $\lambda_i(\Phi(X,Y))$, $\lambda_i(J_h(X))$ $(i = 1,2,3...n)$ are respectively the eigenvalues of matrices $\Psi(X,Y), \Phi(X,Y)$ and $J_h(X)$. If,

$$a_0 b_0 - c_1 > 0, \tag{6}$$

and

$$\eta \leq \min\left\{\frac{b_1 q_1}{b_0}, \frac{b_0(c_0+c_1)}{c_1(1+b_0)}\right\}, \tag{7}$$

then, the zero solution of system (2) is uniformly-asymptotically stable.

Proof. To prove this theorem, we make use of the differentiable scalar function $V(t) = V(X(t), Y(t), Z(t))$ defined as

$$\begin{aligned} 2V(t) = 2\eta(a_0+c_1)\int_0^1 \langle H(\sigma X), X\rangle d\sigma + (1+b_0)\langle Z, Z\rangle + 2\eta(1+b_0)\langle H(X), Y\rangle + 2(c_1+a_0)\langle Y, Z\rangle \\ + 2(b_0+1)q(t)\int_0^1 \langle \sigma\Phi(X,\sigma Y)Y, Y\rangle d\sigma + 2(a_0+c_1)r(t)\int_0^1 \langle \sigma\Psi(X,\sigma Y)Y, Y\rangle d\sigma, \end{aligned} \tag{8}$$

where constants a_0, b_0, c_1 and η are as defined before. By Lemma (2.2) we have,

$$\begin{aligned} \langle H(X), H(X)\rangle &= 2\int_0^1 \langle J_h(\sigma X)X, H(\sigma X)\rangle d\sigma, \\ &= 2\int_0^1\int_0^1 \langle \sigma J_h(\sigma_1\sigma X)X, J_h(\sigma X)X\rangle d\sigma d\sigma_1. \end{aligned} \tag{9}$$

Applying Lemma (2.1) in (9) and then integrate from 0 to 1, gives

$$c_0^2\|X\|^2 \leq 2\int_0^1\int_0^1 \langle \sigma J_h(\sigma_1\sigma X)X, J_h(\sigma X)X\rangle d\sigma d\sigma_1 \leq c_1^2\|X\|^2. \tag{10}$$

Similarly,

$$2\int_0^1 \langle H(\sigma X), X\rangle d\sigma = 2\int_0^1\int_0^1 \sigma\langle J_h(\sigma\sigma_1 X)X, X\rangle d\sigma d\sigma_1. \tag{11}$$

By the application of Lemma (2.1) in (11), follows by integrating between 0 and 1, we have

$$c_0\|X\|^2 \leq 2\int_0^1\int_0^1 \sigma\langle J_h(\sigma\sigma_1 X)X, X\rangle d\sigma d\sigma_1 \leq c_1\|X\|^2. \tag{12}$$

Thus, using (9) and (11) in (8), we obtain

$$\begin{aligned} 2V(t) =& \| a_0Y + Z \|^2 + \eta \| b_0^{\frac{1}{2}}Y + b_0^{-\frac{1}{2}}H(X) \|^2 + \frac{c_1}{b_0}(a_0b_0 - c_1)\|Y\|^2 + \eta \| b_0Y + H(X) \|^2 \\ &+ 2(b_0+1)\int_0^1 \sigma\langle (\Phi(X,\sigma Y)q(t) - Ib_0\eta)Y, Y\rangle d\sigma \\ &+ 2(a_0+c_1)\int_0^1 \sigma\langle (\Psi(X,\sigma Y)r(t) - Ia_0)Y, Y\rangle d\sigma + \| c_1b_0^{-\frac{1}{2}}Y + b_0^{\frac{1}{2}}Z \|^2 \\ &+ \frac{2\eta}{b_0}\int_0^1\int_0^1 \sigma\langle [b_0(c_1 - J_h(\sigma X)) + (a_0b_0 - J_h(\sigma X))]J_h(\sigma\sigma_1 X)X, X\rangle d\sigma d\sigma_1. \end{aligned} \tag{13}$$

Clearly, it can be verified by condition (iii) and inequality (6) of the theorem, that the function $V(t)$ as defined in (8) or (13) is positive definite. Also, by applying the conditions of the theorem to (13), we have

$$\begin{aligned} 2V(t) \geq& \| a_0Y + Z \|^2 + \eta \| b_0^{\frac{1}{2}}Y + b_0^{-\frac{1}{2}}H(X) \|^2 + \eta \| b_0Y + H(X) \|^2 + \| c_1b_0^{-\frac{1}{2}}Y + b_0^{\frac{1}{2}}Z \|^2 \\ &+ \frac{c_0\eta}{b_0}(a_0b_0 - c_1)\|X\|^2 + \frac{c_1}{b_0}(a_0b_0 - c_1)\|Y\|^2. \end{aligned} \tag{14}$$

Since all the coefficients of the terms present in (14) are all positive, then, there exists a positive constant K_1 such that

$$2V(t) \geq K_1(\| X \|^2 + \| Y \|^2 + \| Z \|^2). \tag{15}$$

Similarly, from (13) we have

$$\begin{aligned} 2V(t) \leq & \| a_0 Y + Z \|^2 + \eta \| b_0^{\frac{1}{2}} Y + b_0^{-\frac{1}{2}} H(X) \|^2 + \eta \| b_0 Y + H(X) \|^2 + \| c_1 b_0^{-\frac{1}{2}} Y + b_0^{\frac{1}{2}} Z \|^2 \\ & + \big[(b_0+1)(b_1 q_1 - b_0 \eta) + (a_0 + c_1)(a_1 r_1 - a_0) + c_1 b_0^{-1}(a_0 b_0 - c_1)\big] \|Y\|^2 \\ & + \frac{c_1 \eta}{b_0}\big[b_0(c_1 - c_0) + (a_0 b_0 - c_0)\big] \|X\|^2. \end{aligned} \tag{16}$$

Again, all the coefficients of the terms present in (16) are all positive, we can find a positive constant K_2 such that

$$2V(t) \leq K_2(\| X \|^2 + \| Y \|^2 + \| Z \|^2). \tag{17}$$

Now, suppose that $(X, Y, Z) = (X(t), Y(t), Z(t))$ is any solution of system (2) and then differentiate the function $V(t)$ in (8) with respect to t along the system (2), we get

$$\begin{aligned} \dot{V}(t) = & -(1+b_0)\langle Z, \Psi(X,Y) r(t) Z \rangle + (c_1 + a_0)\langle Z, Z \rangle - (c_1 + a_0)\langle Y, \Phi(X,Y) q(t) Y \rangle \\ & + \eta(b_0+1)\langle J_h(X) Y, Y \rangle + \langle (b_0+1) Z + (c_1 + a_0) Y, P(t, X, Y, Z) \rangle \\ & + (b_0+1) q'(t) \int_0^1 \langle \sigma \Phi(X, \sigma Y) Y, Y \rangle d\sigma + (a_0 + c_1) r'(t) \int_0^1 \langle \sigma \Psi(X, \sigma Y) Y, Y \rangle d\sigma \\ & + \int_0^1 \langle \sigma J(\Psi(X, \sigma Y) Y | X) Y, Y \rangle d\sigma + \int_0^1 \langle \sigma J(\Phi(X, \sigma Y) Y | X) Y, Y \rangle d\sigma. \end{aligned}$$

But since, $r'(t)$, $q'(t)$, $J(\Phi(X, \sigma Y) Y | X)$ and $J(\Psi(X, \sigma Y) Y | X)$ are semi-negative, then,

$$\begin{aligned} \dot{V}(t) \leq & -(1+b_0)\langle Z, \Psi(X,Y) r(t) Z \rangle + (c_1 + a_0)\langle Z, Z \rangle - (c_1 + a_0)\langle Y, \Phi(X,Y) q(t) Y \rangle \\ & + \eta(b_0+1)\langle J_h(X) Y, Y \rangle + \langle (b_0+1) Z + (c_1 + a_0) Y, P(t, X, Y, Z) \rangle. \end{aligned} \tag{18}$$

Setting $P(t, X, Y, Z) = 0$ in (18) and then use the conditions of Theorem (3.1), Lemma (2.1), and Lemma (2.2), we obtain

$$\begin{aligned} \dot{V}(t) & \leq -\big[b_0(c_1 + c_0) - c_1 \eta (1 + b_0)\big] \langle Y, Y \rangle - (a_0 b_0 - c_1)\langle Z, Z \rangle \\ & = -\delta_0 \{ \| Y \|^2 + \| Z \|^2 \} \leq 0, \end{aligned}$$

where $\delta_0 = \min\{(a_0 b_0 - c_1), b_0(c_1 + c_0) - c_1 \eta (1 + b_0)\}$.

Thus, it is clear that,

$$V(X, Y, Z) \to \infty \quad as \quad \| X \|^2 + \| Y \|^2 + \| Z \|^2 \to \infty. \tag{19}$$

Now, let us consider a set defined by

$$Q \equiv \{(X, Y, Z) : \dot{V}(X, Y, Z) = 0\}.$$

By applying the well-known LaSalle's invariance principle, we note that $(X,Y,Z) \in Q$ implies that $Y = Z = 0$. But from system (2) and Lemma (2.2), we have $H(X) = 0$ which implies $X = 0$. Therefore, $X = Y = Z = 0$. This fact shows that the largest invariant set contained in Q is $(0,0,0) \in Q$. Hence, our conclusion is that the trivial solution of equation (1) or (2) is uniformly-asymptotically stable when $P(t,X,Y,Z) \equiv 0$. □

Example 3.1. *We provide the following example as a special case of system (2) with $n = 2$. In system (2), let*

$$\Psi(X,Y) = \begin{pmatrix} 4 + \frac{1}{x_1^2+y_1^2+5} & 1 \\ 1 & 4 + \frac{1}{x_2^2+y_2^2+5} \end{pmatrix}, \quad H(X) = \begin{pmatrix} x_1 + 0.0001\sin x_1 \\ x_2 + 0.0001\sin x_2 \end{pmatrix},$$

$$\Phi(X,Y) = \begin{pmatrix} \frac{5}{2} + \frac{1}{1+x_2^2} & 0 \\ 0 & \frac{5}{2} + e^{-(x_1+y_1)^2} \end{pmatrix}, \eta = \frac{6}{5}, \; r(t) = 1 + e^{-t},$$

and

$$q(t) = 1 + \frac{1}{1+t^2}.$$

Thus, the Jacobian matrix $J_h(X)$ of $H(X)$ is,

$$J_h(X) = \frac{6}{5}\begin{pmatrix} 1 + 0.0001\cos x_1 & 0 \\ 0 & 1 + 0.0001\cos x_2 \end{pmatrix}.$$

It is obvious that Ψ, Φ, and J_h are symmetric and positive definite and by some simple calculations, we obtain their eigenvalues as follows.

$$\lambda_1(\Psi(X,Y)) = 5 + \frac{1}{x_1^2+y_1^2+5}, \quad \lambda_2(\Psi(X,Y)) = 3 + \frac{1}{x_2^2+y_2^2+5},$$

$$\lambda_1(\Phi(X,Y)) = \frac{5}{2} + e^{-(x_1+y_1)^2}, \quad \lambda_2(\Phi(X,Y)) = \frac{5}{2} + \frac{1}{1+x_2^2},$$

$$\lambda_1(J_h(X)) = \frac{6}{5}(1 + 0.0001\cos x_1), \; \textit{and} \; \lambda_2(J_h(X)) = \frac{6}{5}(1 + 0.0001\cos x_2).$$

From the eigenvalues above, we have $a_0 = 3, \; a_1 = 5.2, \; b_0 = 2.5, \; b_1 = 3.5, \; c_0 = 1.19988, \; c_1 = 1.20012$. Also,

$$1 \leq r(t) = 1 + e^{-t} \leq 2$$

and

$$0 \geq r'(t) = -e^{-t}$$

.
Similarly,

$$1 \leq q(t) = 1 + \frac{1}{1+t^2} \leq 2$$

and

$$0 \geq q'(t) = \frac{-2t}{(1+t^2)^2}$$

.
Hence,

$$a_0 b_0 - c_1 = 7.5 - 1.20012 = 6.29988 > 0,$$

and,

$$\begin{aligned}\eta = \frac{6}{5} &\leq \min\left\{\frac{b_1 q_1}{b_0}, \frac{b_0(c_0+c_1)}{c_1(1+b_0)}\right\}\\ &\leq \min\left\{\frac{7}{2.5}, \frac{6}{4.20042}\right\}\\ \frac{6}{5} &< \frac{6}{4.20042}.\end{aligned}$$

Therefore, all the conditions of Theorem (3.1) are satisfied.

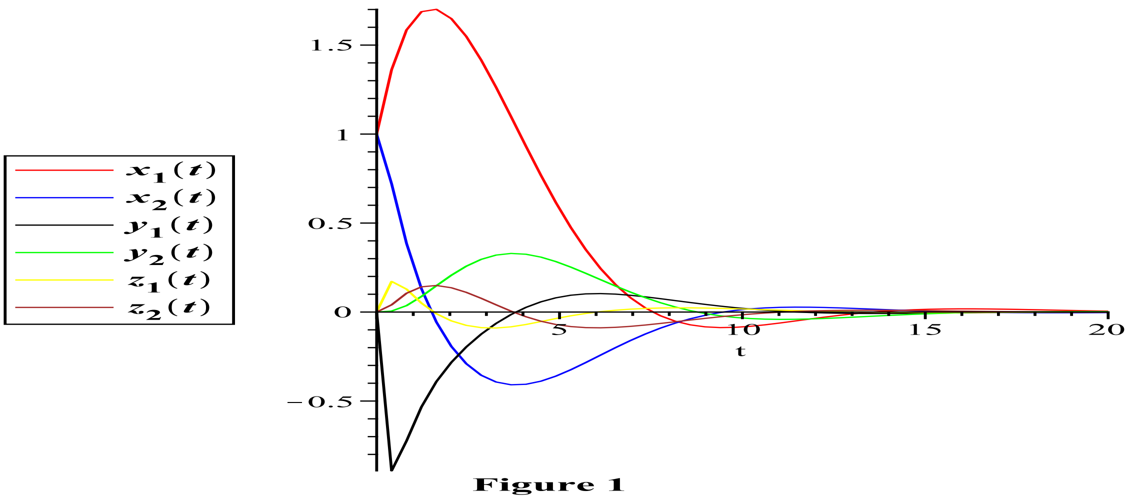

Figure 1. Shows that the trivial solution of the equation in Example (3.1) is uniformly-asymptotically stable even as fast as $t \to 20$.

4. Boundedness result

Now, for the case $P(t, X, Y, Z) \neq 0$ we have the following theorem.

Theorem 4.1. *Suppose in addition to the hypotheses of Theorem (3.1), there exists a non-negative continuous function* $e(t)$*, with*

$$\| P(t, X, Y, Z) \| \leq e(t), \tag{20}$$

for all $t \geq 0$*,* $\max e(t) < \infty$ *and* $e(t) \in L^1(0, \infty)$ *(i.e. the space of integrable Lebesque functions). Then, there exists a positive constant* D *such that any solution* $(X(t), Y(t), Z(t))$ *of system (2) determined by*

$$X(0) = X_0,\ Y(0) = Y_0,\ Z(0) = Z_0,$$

satisfies

$$\| X(t) \| \leq D, \quad \| Y(t) \| \leq D, \quad \| Z(t) \| \leq D, \tag{21}$$

for all $t \in \mathbb{R}^+$.

Proof. In proving this theorem, we make use of the Lyapunov function defined in (6). It should be noted that the estimate (15) and (17) in the proof of Theorem (3.1) are still valid when $P(t,X,Y,Z) \neq 0$. So, from (18), we have

$$\begin{aligned}\dot{V}(t) &\leq -\delta_0\{\| Y \|^2 + \| Z \|^2\} + \langle\{(a_0+c_1)Y + (1+b_0)Z\}, P(t,X,Y,Z)\rangle \\ &\leq \langle(a_0+c_1)Y + (1+b_0)Z, P(t,X,Y,Z)\rangle \\ &\leq \delta_2\{\| Y \| + \| Z \|\} \| P(t,X,Y,Z) \|, \end{aligned} \quad (22)$$

where

$$\delta_2 = \max\{(1+b_0); (a_0+c_1)\}.$$

Now, by applying condition (20) of Theorem (4.1) and the fact that $\|Y\| \leq 1 + \|Y\|^2$ and $\|Z\| \leq 1 + \|Z\|^2$ to (22), we obtain

$$\begin{aligned}\dot{V}(t) &\leq \delta_2 e(t)\{\| Y \| + \| Z \|\} \\ &\leq \delta_2 e(t)\{2 + \| Y \|^2 + \| Z \|^2\} \\ &= 2\delta_2 e(t) + \delta_2 e(t)\{\| Y \|^2 + \| Z \|^2\}.\end{aligned}$$

From (15), we have

$$\begin{aligned}\dot{V}(t) &\leq 2\delta_2 e(t) + \delta_2 K_1^{-1} e(t) V(t) \\ &\leq \delta_5 e(t) + \delta_6 e(t) V(t),\end{aligned} \quad (23)$$

where $\delta_5 = 2\delta_2$ and $\delta_6 = \delta_2 K_1^{-1}$. Integrating (23) from 0 to t, we get

$$V(t) - V(0) \leq \delta_5 \int_0^t e(s)ds + \delta_6 \int_0^t e(s)V(s)ds$$

$$V(t) \leq V(0) + \delta_5 \int_0^t e(s)ds + \delta_6 \int_0^t e(s)V(s)ds$$

$$V(t) \leq \delta_7 + \delta_6 \int_0^t e(s)V(s)ds,$$

with $\delta_7 = V(0) + \delta_5 \int_0^t e(s)ds$.

By applying Gronwall-Bellman inequality [21], we have

$$V(t) \leq \delta_7 \exp(\delta_6 \int_0^t e(s)ds) \leq D_1,$$

for some positive constant D_1. This in turn implies that

$$\| X(t) \| \leq D_1, \quad \| Y(t) \| \leq D_1, \quad \| Z(t) \| \leq D_1.$$

Thus, the proof of Theorem (4.1) is completes on taking $D_1 = D$. □

Example 4.1. *Given that in addition to the Example (3.1), we define*

$$P(t, X, Y, Z) = \frac{1}{1+t^2}\begin{pmatrix} 2 + \frac{1}{x^2+y^2+1} \\ \exp^{-(z^2+y^2)} \end{pmatrix}.$$

By some simple calculations, we can show that

$$\begin{aligned} \| P(t, X, Y, Z) \| &\leq \frac{\sqrt{10}}{1+t^2} \\ &= e(t). \end{aligned}$$

$$\textit{such that } \max e(t) = \sqrt{10} < \infty, \textit{ and } \int_0^\infty \frac{\sqrt{10}}{1+t^2} dt = \frac{\sqrt{10}\pi}{2},$$

which means

$$e(t) \in L^1(0, \infty).$$

Thus, all the conditions of Theorem (4.1) are satisfied.

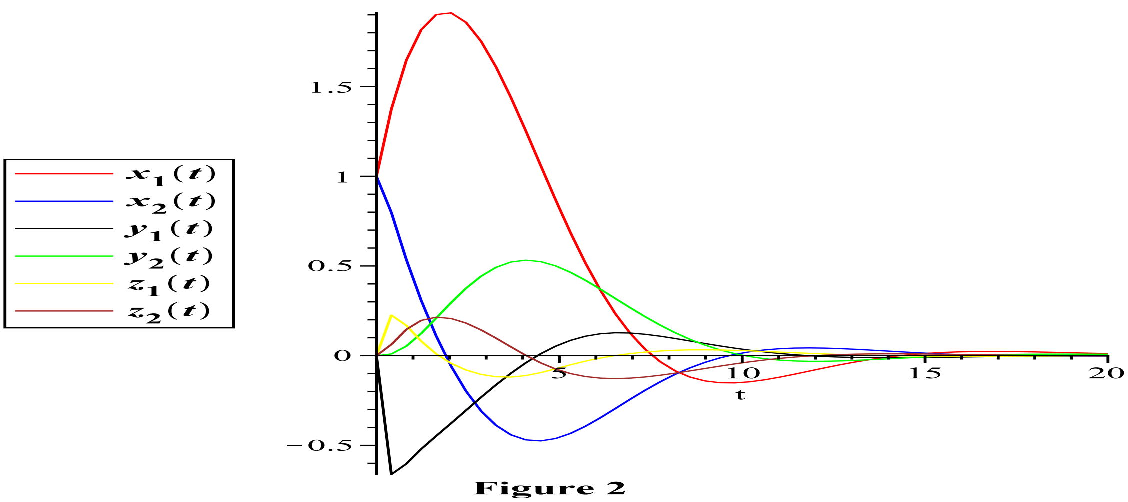

Figure 2. Demonstrates the boundedness of all solutions of the equation in Example (4.1) when $P(t, X, Y, Z) \neq 0$.

Remark 4.1.
Theorem (3.1) and Theorem (4.1) are still valid for the scalar differential equation (5). Hence, with some simple modifications, the stability and boundedness problem of equation (5) are easily addressed.

5. Conclusion

In this paper, asymptotic stability of the trivial solution when $P(t, X, Y, Z) = 0$ and boundedness of all solutions when $P(t, X, Y, Z) \neq 0$ of a class of third order differential equation are considered. Our results are established using the direct method of Lyapunov and they improved and generalized many existing results in literature especially those of ([17] and [25]). It is evident from Figure 1 and Figure 2 obtained by Maple software, that the simulated solutions of the examples constructed are stable and bounded respectively. Uniform ultimate boundedness of all solutions of equation (1) will be considered in the future research if we are able to come up with a complete Lyapunov function. Similarly, the presents results can be generalized to delay stochastic differential equation as well, provided a suitable Lyapunov functional can be constructed.

Conflict of Interest
There is no conflict of interest about this work.
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