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The Chatterjea fixed point theorem on CCRM - spaces

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Abstract: In the present paper, we obtain a fixed point result on CCRM (Complete Cone Rectangular) - Spaces. We extend the results of Jleli and samet results existing in the literature.

Key words: Cone metric space, cone rectangular metric space, fixed point, normal cone.
AMS subject classifications: 47H10, 54H25.

1. Introduction and Preliminaries

Banach fixed point theorem is first and foremost one in fixed point theory and lot of generalizations has been done in this theorem. Huang and Zhang [5] have introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space and they obtained fixed point results for contractive type conditions in normal cone metric space. Subsequently many authors has been studying and generalizing this cone metric space [see for e.g. [1–16]. Branciari [3], Azam, Arshad and Beg [1] extended the notion of cone metric spaces by replacing the triangular inequality by a rectangular inequality. Recently. Jleli and Samet [4] obtained a fixed point theorem in a cone rectangular metric space. In this paper, we have generalized and extended the results of [4].

We need some of preliminary definitions in our main result which are due to [4].

Definition 1.1. Let M always be a real Banach space and P a subset of M. Q is called a cone if and only if:

(a). Q is closed, non empty, and $Q \neq 0$.

(b). $\alpha, \beta \in R,\ \alpha, \beta \geq 0,\ x, y \in Q$ *implies* $\alpha x + \beta y \in Q$.

(c). $x \in Q$ and $-x \in Q$ implies $x = 0$.

Definition 1.2. Given a cone $Q \subset M$, we define a partial ordering $\leq$ with respect to Q by : $x \leq y$ iff $y - x \in Q$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in$ interior of Q.

Definition 1.3. The cone Q is called normal if there is a number $L > 0$ such that for all $x, y \in M$, $0 \leq x \leq y$ implies $\|x\| \leq L\|y\|$ where $\|.\|$ is the norm in M. In this case the number L is called the normal constant of Q.

In the following we always suppose M is a Banach space, Q is a cone in M with int $Q \neq \phi$ and $\leq$ is partial ordering with respect to Q.

Definition 1.4. Let X be a non- empty set. Suppose the $\rho : X \to X$ satisfies the following:

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(a). $\rho(x,y) > 0$ for all $x, y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$.

(b). $\rho(x,y) = \rho(y,x)$, *for all* $x, y \in X$.

(c). $\rho(x,y) \leq \rho(x,w) + \rho(w,z) + \rho(z,y)$, for all $x, y \in X$ and for all distinct points $w, z \in X - x, y$ [rectangular property].

Then ρ is called a cone rectangular metric on X, and (X,ρ) is called a cone rectangular metric space.

Definition 1.5. Let (X,ρ) be a cone rectangular metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $d \in M$, $d >> 0$ there is N such that for all $n > N$, $\rho(x_n, x) << d$, then (x_n) is said to be convergent to x and x is the limit of (x_n). We denote this $x_n \to x$ as $n \to \infty$.

Definition 1.6. Let (X,ρ) be a cone rectangular metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $d \in M$, $d >> 0$ there is N such that for all $n, m > N$, $\rho(x_n, x_m) << d$, then (x_n) is said to be Cauchy sequence in X.

Definition 1.7. Let (X,ρ) be a cone rectangular metric space. If every Cauchy sequence is convergent, then X is called a CCRM(Complete Cone Rectangular Metric)-space.

Lemma 1.1. *Let (X,ρ) be a cone rectangular metric space and M be a normal cone. Let (x_n) be a sequence in X. Then (x_n) Cauchy sequence in X if and only if, $\rho(x_n, x_m) \to 0$ as $m, n \to \infty$.*

2. Main Result

Theorem 2.1. *Let (N,ρ) be a CCRM-Space, S be a normal cone with normal constant L. Suppose a mapping $A : N \to N$ satisfies the following contraction condition*

$$\rho(Ax, Ay) \leq a[\rho(x, Ay) + \rho(y, Ax)]. \tag{1}$$

For all $x, y \in X$, where $a \in [0, 1/2)$. Then

(i). A has a fixed point in X.

(ii). for any $x \in X$ the iterative sequence $\{A^n x\}$ converges to the fixed point.

Proof. Let $x \in N$, we have

$$\begin{aligned} \rho(Ax, A^2x) &\leq a[\rho(x, A^2x) + \rho(Ax, Ax)], \\ &\leq a\rho(x, A^2x), \\ &\leq a[\rho(x, Ax) + \rho(Ax, A^2x)], \\ &\leq \frac{a}{1-a}\rho(x, Ax). \end{aligned}$$

Again

$$\begin{aligned} \rho(A^2x, A^3x) &\leq a[\rho(Ax, A^3x) + \rho(A^2x, A^2x)], \\ &\leq a\rho(Ax, A^3x), \\ &\leq a[\rho(Ax, A^x) + \rho(A^2x, A^3x)], \end{aligned}$$

$$\leq \frac{a}{1-a}\rho(Ax, A^2x).$$
$$\leq [\frac{a}{1-a}]^2\rho(Ax, x).$$

Then in general n is positive integer,

$$\rho(A^2x, A^3x) \leq [\frac{a}{1-a}]^n\rho(Ax, x).$$
$$\leq h^n\rho(Ax, x), \text{ where } h = \frac{a}{1-a} \in [0,1).$$

Divide the proof into two cases:

Case-I: Let $A^m x = A^n x$ for some $m, n \in \mathbb{N}, m \neq n$. Let $m > n$, then

$A^{m-n}(A^n x) = A^n x$, that is, $A^p y = y$, where $p = m - n$, $y = A^n x$.

Now since $p > 1$ we have

$$\rho(y, Ay) = \rho(A^p x, A^{p+1}x) \leq h^p\rho(y, Ay).$$

Since $h \in [0,1)$, we obtain

$$-\rho(y, Ay) \in P \text{ and } \rho(y, Ay) \in P.$$

Implies that $\|\rho(y, Ay)\| = 0$. That is, $y = Ay$.

Case-II: Assume that $A^m x \neq A^n x$ for some $m, n \in N$, $m \neq n$. Let $m > n$, clearly we have

$$\rho(A^n x, A^{n+1}x) \leq h^p\rho(x, Ax)$$
$$\leq \frac{h^n}{1-h}\rho(x, Ax),$$

and

$$\begin{aligned}
\rho(A^n x, A^{n+2}x) &\leq a[\rho(A^{n-1}x, A^{n+2}x) + \rho(A^{n+1}x, A^n x)], \\
&= a[\rho(A^{n-1}x, A^n x) + \rho(A^{n+1}x, A^n x)], \\
&= a[\rho(A^{n-1}x, A^n x) + \rho(A^n x, A^{n+2}x) + \rho(A^{n+1}x, A^n x)], \\
&= a[h^n\rho(x, Ax) + \rho(A^n x, A^{n+1}x) + \rho(A^{n+1}x, A^{n+2}x) + h^n\rho(Ax, x)], \\
&= a[h^{n-1}\rho(Ax, x) + h^n\rho(Ax, x) + h^{n+1}\rho(Ax, x) + h^n\rho(Ax, x)], \\
&\leq a[h^{n-1}\rho(Ax, x) + 2h^n\rho(Ax, x) + h^{n+1}\rho(Ax, x)], \\
&\leq h^n\rho(Ax, x) + 2h^n\rho(Ax, x) + h^{n+1}\rho(Ax, x), \\
&\leq 3h^n\rho(Ax, x) + h^{n+1}\rho(Ax, x), \\
&\leq \frac{h^n}{1-h}\rho(x, Ax).
\end{aligned}$$

If $m > 2$ is odd then write $m = 2l + 1$, $l > 1$ and using the fact that $A^p x \neq A^h x$ for $p, h \in N$, $p \neq h$, we can easily show that

$$
\begin{aligned}
\rho(A^n x, A^{n+m} x) &\leq \rho(A^n x, A^{n+1} x) + \rho(A^{n+1} x, A^n x) + \rho(A^{n+1} x, A^{n+2} x) + \cdots + \rho(A^{n+2l} x, A^{n+2l+1} x)], \\
&\leq h^n \rho(x, Ax) + h^{n+1} \rho(x, Ax) + \ldots + h^{n+2l} \rho(x, Ax), \\
&\leq h^n [1 + h + h^2 + \ldots] \rho(x, Ax), \\
&\leq \frac{h^n}{1-h} \rho(x, Ax).
\end{aligned}
$$

Again if $m > 2$ is even then written as $m = 2l$, $l \geq 2$ and using the same arguments as before we get that

$$
\begin{aligned}
\rho(A^n x, A^{n+m} x) &\leq \rho(A^n x, A^{n+2} x) + \rho(A^{n+2} x, A^{n+3} x) + \cdots + \rho(A^{n+2l-1} x, A^{n+2l} x), \\
&\leq h^n \rho(x, Ax) + h^{n+2} \rho(x, Ax) + \ldots + h^{n+2l} \rho(x, Ax), \\
&\leq h^n [1 + h + h^2 + \ldots] \rho(x, Ax), \\
&\leq \frac{h^n}{1-h} \rho(x, Ax).
\end{aligned}
$$

Then combining all the above cases we have

$$
\rho(A^n x, A^{n+m} x) \leq \frac{h^n}{1-h} \rho(x, Ax) \text{ forall } m, n \in \mathbb{N}.
$$

Hence we get that

$$
\|\rho(A^n x, A^{n+m} x)\| \leq K \frac{h^n}{1-h} \|\rho(x, Ax)\| \text{ forall } m, n \in N.
$$

Since

$$
K \frac{h^n}{1-h} \|\rho(x, Ax)\| \to 0 \text{ as } n \to \infty
$$

Claim: $Ax^* = x^*$. Without loss of generality we assume that $Ax^* \neq x^*$, for any $n \in N$, we have

$$
\begin{aligned}
\rho(x^*, Ax^*) &\leq \rho(x^*, A^n x) + \rho(A^n x, A^{n+1} x) + \rho(A^{n+1} x, Ax^*), \\
&\leq \rho(x^*, A^n x) + \rho(A^n x, A^{n+1} x) + \rho(A^{n+1} x, Ax^*), \\
&\leq \rho(x^*, A^n x) + \rho(A^n x, A^{n+1} x) + a[\rho(A^n x, Ax^*) + \rho(x^*, A^{n+1} x^*)] \\
\rho(x^*, Ax^*) &\leq 0.
\end{aligned}
$$

Therefore, $\rho(x*, Ax^*) = 0$.

That implies $x^* = Ax^*$. Therefore A has a fixed point. □

3. Conclusion

In this paper, our results are extended and generalized results of [4].

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