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Research on neighborhood-based characterization and separation axioms of (L, M)-fuzzy convex spaces

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Abstract: This study focuses on the construction of concave (L, M)-fuzzy neighborhood operators and a systematic analysis of the separation axiom system in (L, M)-fuzzy convex spaces. First, concave (L, M)-fuzzy neighborhood operators are formally defined, and a bidirectional induction mechanism between these operators and (L, M)-fuzzy convex structures is established via (L, M)-fuzzy concave structures. The isomorphism between the category of (L, M)-fuzzy convex spaces and that of concave (L, M)-fuzzy neighborhood spaces is proved, establishing a theoretical connection between local neighborhood characterizations and global convex structures. Second, in the study of separation axioms, the validity of S_3 and S_4 axioms in (L, M)-fuzzy convex spaces is verified, particularly the hereditary property of S_3 in subspaces and, under certain conditions, the preservation of the S_3 axiom under products are established, enriching the property system of separation axioms within the framework of fuzzy convexity.

Key words: (L, M)-fuzzy convex spaces, concave (L, M)-fuzzy neighborhood operators, subspace and product of (L, M)-fuzzy convex spaces, S_3 and S_4 separation axioms, isomorphic categories

1. Introduction

Abstract convexity theory [23, 28], a subfield of mathematics, boasts extensive connections with other mathematical disciplines and has found applications in diverse research fields, including metric spaces, topological spaces, graphs, and lattices (see, e.g., [5, 11, 19, 24, 27, 29]). With the development of fuzzy mathematics, fuzzy set theory has been integrated into numerous mathematical structures—such as fuzzy convergence structures [6, 12] and fuzzy topology [4, 25, 26, 32].

In the context of generalizing convex structures, Rosa [19] first introduced the concept of fuzzy convex structures in 1994, sparking subsequent research on this structure by various scholars. In 2009, Maruyama [10] further generalized convex structures to propose a new type, namely L-convex structures; within this theoretical framework, Pang [15, 16] later identified several key characteristics of L-convex structures. In 2017, Shi and Xiu [21] advanced this line of work by introducing a more general form of fuzzifying convex structures, referred to as (L, M)-fuzzy convex structures, which encompasses the aforementioned two types (i.e., L-convex structures and M-fuzzifying convex structures) as special cases. To date, researchers have explored fuzzy convex structures from various perspectives, including bases and subbases, fuzzy hull operators, fuzzy interval operators, fuzzy interior operators, product and coproduct structures, fuzzy betweenness relations, and fuzzy remote neighborhood operators (see, e.g., [13, 14, 31, 33–37, 39]).

Separation axioms are fundamental to convex structure theory. Jamison [3] first introduced them in

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this context and proposed a restricted polytope screening characterization using half-space-based screening. Rosa [18] later extended the definition to L-convex structures. However, prior to our work, separation axioms had not been defined for (L, M)-fuzzy convex structures. Motivated by this gap, Zhao et al. [38] defined them for (L, M)-fuzzy convex structures using (L, M)-fuzzy hull operators and r-L-fuzzy biconvex sets from Sayed et al. [20]. Liang et al. [9] further introduced S_i $(i = 0, 1, 2)$ separation axioms in (L, M)-fuzzy convex spaces but did not extend them to S_3 and S_4. Building on this unresolved issue and the lack of a neighborhood-based characterization, this paper aims to investigate S_3 and S_4 separation axioms and establish such a characterization for (L, M)-fuzzy convex spaces.

The organization of this paper is as follows. In Section 2, we review the necessary concepts and notations. Next, in Section 3, the concept of concave (L, M)-fuzzy neighborhood operators is introduced, and the relationship between concave (L, M)-fuzzy neighborhood operators and (L, M)-fuzzy concave structures is investigated. In Section 4, the notions of S_3 and S_4 separation axioms for (L, M)-fuzzy convex spaces are defined. The relationships among various separation axioms are discussed, with particular focus on their hereditary properties in subspaces and the product property of the S_3 separation axiom. Finally, Section 5 summarizes the main research findings and conclusions of this work.

2. Preliminaries

Throughout this paper, let M denote a completely distributive lattice with least and greatest elements denoted by $\mathbf{0}_M$ and $\mathbf{1}_M$, respectively. For $a, b \in M$, we say a is wedge below b in M (cf. [17, 30]), denoted $a \prec b$, if for every subset $D \subseteq M$, whenever $b \leq \bigvee D$, there exists some $d \in D$ such that $a \leq d$. We define $\beta(x) = \{y \in M \mid y \prec x\}$. A complete lattice M is completely distributive if and only if for each $x \in M$, $x = \bigvee \beta(x) = \bigvee \beta^*(x)$, where $\beta^*(x) = \beta(x) \cap J(M)$ is termed the standard greatest minimal family of x, and $J(M)$ denotes the set of non-zero coprime elements in M (cf. [30]).

Let $M_{\mathbf{0}_M} = M \setminus \{\mathbf{0}_M\}$, where $\mathbf{0}_M$ is the least element of M. In a completely distributive lattice M, the implication $\rightarrow: M \times M \rightarrow M$ (as the right adjoint of the meet operation $\wedge$) is defined by

$$a \rightarrow b = \bigvee \{c \in M \mid a \wedge c \leq b\}.$$

Lemma 2.1. *[2] Let $(M, \vee, \wedge)$ be a completely distributive lattice and $\rightarrow$ the implication operation induced by $\wedge$. For all $a, b, c \in M$ and families $\{a_i\}_{i \in I}$, $\{b_i\}_{i \in I} \subseteq M$, the following hold:*

(1) $(a \rightarrow b) \geq c \iff a \wedge c \leq b$;

(3) $a \leq b \iff a \rightarrow b = \top$;

(3) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$;

(4) $(c \rightarrow a) \wedge (a \rightarrow b) \leq c \rightarrow b$;

(5) $c \rightarrow a \leq (a \rightarrow b) \rightarrow (c \rightarrow b)$;

(6) $a \rightarrow \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} (a \rightarrow a_i)$, *and thus* $a \rightarrow b \leq a \rightarrow c$ *whenever* $b \leq c$;

(7) $\bigvee_{i \in I} a_i \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b)$, *and thus* $a \rightarrow c \geq b \rightarrow c$ *whenever* $a \leq b$.

In this paper, let L denote a completely distributive lattice endowed with an order-reversing involution "$\prime$". An element $a \in L$ is called coprime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of non-zero coprime elements in L is denoted by $J(L)$.

For a nonempty set X, let L^X denote the set of all L-fuzzy subsets of X. The set L^X forms a completely distributive lattice under the point-wise order. The smallest and largest elements in L^X are denoted by $\mathbf{0}_X$ and $\mathbf{1}_X$, respectively. It can be verified that the set of non-zero coprime elements in L^X is $\{x_\lambda \mid x \in X, \lambda \in J(L)\}$, where x_λ is a non-zero coprime L-fuzzy subset of X parameterized by $\lambda \in J(L)$.

Definition 2.1. [1] A mapping $\mathcal{C} : L^X \to M$ is called an (L, M)-fuzzy closure system on X if it satisfies the following:

(1) $\mathcal{C}(\mathbf{0}_X) = \mathcal{C}(\mathbf{1}_X) = 1_M$;

(2) For any nonempty family $\{B_i : i \in J\} \subseteq L^X$, we have $\mathcal{C}\left(\bigwedge_{i\in J} B_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(B_i)$.

If $\mathcal{C}$ is an (L, M)-fuzzy closure system on X, the pair $(X, \mathcal{C})$ is referred to as an (L, M)-fuzzy closure space.

Definition 2.2. [14] A closure system $\mathcal{C}$ (in the sense of Definition 2.1) is called an (L, M)-fuzzy convex structure if it satisfies one of the following conditions, where the second is a consequence of the first:

(3) If $\{B_i : i \in J\} \subseteq L^X$ is totally ordered, then $\mathcal{C}\left(\bigvee_{i\in J} B_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(B_i)$.

(3)* If $\{B_i : i \in J\} \subseteq L^X$ is directed, then $\mathcal{C}\left(\bigvee^d_{i\in J} B_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(B_i)$.

If $\mathcal{C}$ is an (L, M)-fuzzy convex structure on X, the pair $(X, \mathcal{C})$ is referred to as an (L, M)-fuzzy convex space.

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be (L, M)-fuzzy convex spaces, and let $g : X \to Y$ be a mapping. We say g is (L, M)-convexity-preserving ((L, M)-CP, for short) if $\mathcal{C}_Y(S) \leq \mathcal{C}_X\left(g^{\leftarrow}(S)\right)$ for all $S \in L^Y$.

It is straightforward to verify that all (L, M)-fuzzy convex spaces (as objects) and all (L, M)-CP mappings (as morphisms) form a category, denoted by (L, M)-**FC**.

Definition 2.3. [21] Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and $Y \subseteq X$ a subset. Then $(Y, \mathcal{C}|_Y)$ is called a subspace of $(X, \mathcal{C})$. More precisely, for each $A \in L^Y$, $(\mathcal{C}|_Y)(A) = \bigvee\left\{\mathcal{C}(B) : B \in L^X, B|_Y = A\right\}$ where $B|_Y$ denotes the restriction of B to Y.

Definition 2.4. [9] Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and $A \in L^X$. The degree $\mathcal{H}_{\mathcal{C}}(A)$ to which A is a biconvex set (half-space, hemispace) is defined by $\mathcal{H}_{\mathcal{C}}(A) = \mathcal{C}(A) \wedge \mathcal{C}(A')$.

Definition 2.5. [13] Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space. A mapping $\mathcal{B} : L^X \to M$ is called a base of $(X, \mathcal{C})$ if it satisfies the following condition: $\mathcal{C}(A) = \bigvee_{\bigvee^d_{i\in I} A_i = A} \bigwedge_{i\in I} \mathcal{B}(A_i)$ for all $A \in L^X$, where $\bigvee^d$ denotes the directed join (i.e., the join of a directed family) in L^X.

Definition 2.6. [13] Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space. A mapping $\mathcal{G} : L^X \to M$ is called a subbase of $(X, \mathcal{C})$ if the mapping $\mathcal{B}_{\mathcal{G}} : L^X \to M$ defined by $\mathcal{B}_{\mathcal{G}}(A) = \bigvee_{\bigwedge_{i\in I} A_i = A} \bigwedge_{i\in I} \mathcal{G}(A_i)$ for all $A \in L^X$ is a base of $(X, \mathcal{C})$.

Definition 2.7. [21] Let $\{(X_i, \mathcal{C}_i)\}_{i\in I}$ be a family of (L, M)-fuzzy convex spaces, $X = \prod_{i\in I} X_i$ and let $p_i : X \to X_i$ be the projection mapping for each $i \in I$. Define a mapping $\mathcal{G} : L^X \to M$ as follows: for each $A \in L^X$, $\mathcal{G}(A) = \bigvee_{i\in I} \bigvee_{p_i^{\leftarrow}(B)=A} \mathcal{C}_i(B)$.

The product (L,M)-fuzzy convex structure $\mathcal{C}$ on X is the structure generated by the subbase $\mathcal{G}$. The resulting (L,M)-fuzzy convex space $(X,\mathcal{C})$ is called the product of $\{(X_i,\mathcal{C}_i)\}_{i\in I}$ and is denoted by $(\prod_{i\in I} X_i, \prod_{i\in I}\mathcal{C}_i)$.

From Definition 2.5 to 2.7, the following product expression can be derived.

Theorem 2.1. *[13] Let $\{(X_j,\mathcal{C}_j) : j\in J\}$ be a set of (L,M)-fuzzy convex spaces, $X=\prod\limits_{j\in J} X_j$, and $p_j : X \longrightarrow X_j$ the projection for each $j\in J$. Then, for each $A\in L^X$,*

$$\Big(\prod_{j\in J}\mathcal{C}_j\Big)(A) \;=\; \bigvee_{\bigvee\limits_{k\in K}^{d} A_k=A}\ \bigwedge_{k\in K}\ \bigvee_{\bigwedge\limits_{i\in I_k} A_{ki}=A_k}\ \bigwedge_{i\in I_k}\ \bigvee_{j\in J}\ \bigvee_{p_j^{\leftarrow}(B^j)=A_{ki}} \mathcal{C}_j(B^j).$$

Definition 2.8. [21] Let $\{(X_i,\mathcal{C}_i)\}_{i\in I}$ be a family of (L,M)-fuzzy convex spaces, and let $(X,\mathcal{C})$ be the product of $\{(X_i,\mathcal{C}_i)\}_{i\in I}$. For each $i\in I$, the projection mapping $p_i : X\to X_i$ is an (L,M)-CP function. Furthermore, $\mathcal{C}$ is the coarsest (L,M)-fuzzy convex structure on X such that all projection mappings $\{p_i : i\in I\}$ are (L,M)-CP functions.

Definition 2.9. [9] Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space. The degree of S_0-separation in $(X,\mathcal{C})$ is defined as

$$S_0(X,\mathcal{C}) = \bigwedge_{\substack{a,b\in J(L^X)\\ a\not\leq b}} \left(\bigvee_{\substack{A\in L^X\\ a\not\leq A \text{ and } A\geq b}} \mathcal{C}(A) \vee \bigvee_{\substack{B\in L^X\\ a\leq B \text{ and } B\not\geq b}} \mathcal{C}(B) \right).$$

Definition 2.10. [8, 9] Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space. The degree of S_1-separation in $(X,\mathcal{C})$ is defined as $S_1(X,\mathcal{C}) = \bigwedge_{a\in J(L^X)} \mathcal{C}(a)$.

When $L=\{0,1\}$, the (L,M)-fuzzy convex space $(X,\mathcal{C})$ reduces to an M-fuzzifying convex space, and the S_1-separation in $(X,\mathcal{C})$ simplifies to $S_1(X,\mathcal{C}) = \bigwedge_{x\in X}\mathcal{C}(\{x\})$.

Definition 2.11. [8, 9] Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space. The degree of S_2-separation in $(X,\mathcal{C})$ is defined as $S_2(X,\mathcal{C}) = \bigwedge_{\substack{a,b\in J(L^X)\\ a\not\leq b}} \bigvee_{\substack{A\in L^X\\ a\not\leq A\geq b}} \mathcal{H}_{\mathcal{C}}(A)$.

When $L=\{0,1\}$, the (L,M)-fuzzy convex space $(X,\mathcal{C})$ reduces to an M-fuzzifying convex space, and the S_1-separation in $(X,\mathcal{C})$ simplifies to $S_2(X,\mathcal{C}) = \bigwedge_{x\neq y} \bigvee_{\substack{A\in 2^X\\ x\in A,\ y\notin A}} \mathcal{H}_{\mathcal{C}}(A)$.

Definition 2.12. [8] For an M-fuzzifying convex space $(X,\mathcal{C})$, we define the degree to which $(X,\mathcal{C})$ is S_3 separated as follows: $S_3(X,\mathcal{C}) = \bigwedge_{A\subseteq X}\bigwedge_{x\notin A}\Big(\mathcal{C}(A) \to \bigvee\limits_{x\notin B\supseteq A} \mathcal{H}_{\mathcal{C}}(B)\Big)$.

Definition 2.13. [8] For an M-fuzzifying convex space $(X,\mathcal{C})$, we define the degree to which $(X,\mathcal{C})$ is S_4 separated as follows: $S_4(X,\mathcal{C}) = \bigwedge_{A\cap B=\emptyset}\Big((\mathcal{C}(A)\wedge\mathcal{C}(B)) \to \bigvee\limits_{A\subseteq C, B\subseteq X\setminus C} \mathcal{H}_{\mathcal{C}}(C)\Big)$.

Definition 2.14. [31] A mapping $\mathcal{A} : L^X \to M$ is an (L,M)-fuzzy concave *structure* on X if it satisfies:

(1) $\mathcal{A}(\mathbf{1}_X) = \mathcal{A}(\mathbf{0}_X) = 1_M$;

(2) For any nonempty family $\{A_i\}_{i\in I} \subseteq L^X$, $\mathcal{A}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathcal{A}(A_i)$;

(3) For any co-directed family $\{A_i \mid i \in I\} \subseteq L^X$ (denoted $\{A_i\}_{i\in I} \overset{\text{codir}}{\subseteq} L^X$), $\mathcal{A}\big(\bigwedge_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathcal{A}(A_i)$.

If $\mathcal{A}$ is an (L,M)-fuzzy concave structure on X, the pair $(X,\mathcal{A})$ is referred to as an (L,M)-fuzzy concave *space*.

Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be (L,M)-fuzzy concave spaces, and let $g: X \to Y$ be a mapping. If $\mathcal{A}\big(g^{\leftarrow}(A)\big) \geq \mathcal{B}(A)$ holds for all $A \in L^Y$, then g is called an (L,M)-fuzzy concavity-preserving function (abbreviated as (L,M)-CAP).

The category with (L,M)-fuzzy concave spaces as objects and (L,M)-CAPs as morphisms is denoted by (L,M)-**FA**.

Theorem 2.2. *[31] The categories (L,M)-**FC** and (L,M)-**FA** are isomorphic.*

In order to clarify the notation used throughout the following deffnitions, theorems and examples, we provide in Table 1 a comprehensive "Symbols and Definitions" summary.

Table 1. Symbols and Definitions Used in the Text

Notation	**Meaning/Definition**
L	Lattice of fuzzy membership degrees
M	Lattice of fuzzy values
$\mathbf{0}_L, \mathbf{1}_L$	Minimum and maximum elements of lattice L
$\mathbf{0}_M, \mathbf{1}_M$	Minimum and maximum elements of lattice M
$J(L), J(M)$	Sets of non-zero coprime elements in lattices L and M
x_λ	Fuzzy point with membership degree λ at point x
$\prec$	$a \prec b$ means a is wedge below b: $\forall D \subseteq M,\ b \leq \bigvee D \Rightarrow \exists d \in D,\ a \leq d$
$\beta(x),\ \beta^*(x)$	$\beta(x) = \{y \in M \mid y \prec x\}, \beta^*(x) = \beta(x) \cap J(M)$
L^X	Family of L-fuzzy subsets on X
$\mathcal{A}: L^X \to M$	(L,M)-fuzzy concave structure
$\mathcal{C}: L^X \to M$	(L,M)-fuzzy convex structure operator
$(X,\mathcal{C})$	(L,M)-fuzzy convex space
$\mathcal{H}_{\mathcal{C}}(A)$	Degree to which A is a bi-convex set: $\mathcal{H}_{\mathcal{C}}(A) = \mathcal{C}(A) \wedge \mathcal{C}(A')$
$\mathcal{N}: L^X \times M_{\mathbf{0}_M} \to L^X$	Concave (L,M)-fuzzy neighborhood operator on X
$\mathcal{N}_{\mathcal{A}}: L^X \times M_{\mathbf{0}_M} \to L^X$	Concave (L,M)-fuzzy neighborhood operator on X
$\mathcal{A}_{\mathcal{N}}: L^X \to M$	(L,M)-fuzzy concave structure on X
$S_i(X,\mathcal{C})$ $(i = 0,1,2,3,4)$	S_i-separation degree in the (L,M)-fuzzy convex space

3. Concave (L,M)-fuzzy neighborhood operators

In this section, we formalize the concept of concave (L,M)-fuzzy neighborhood operators and investigate their interplay with (L,M)-fuzzy concave structures.

We commence by defining concave (L,M)-fuzzy neighborhood operators as follows.

Definition 3.1. Let X be a non-empty set. A mapping $\mathcal{N}: L^X \times M_{0_M} \to L^X$ is called a concave (L,M)-fuzzy neighborhood operator on X if it satisfies the following conditions (where $\forall x \in X$, $A, B \in L^X$, and $a, b \in M_{0_M}$):

(1) $\mathcal{N}(1_X, a)(x) = 1_L$;

(2) $\mathcal{N}(A, a)(x) \leq A(x)$;

(3) If $A \leq B$, then $\mathcal{N}(A, a)(x) \leq \mathcal{N}(B, a)(x)$;

(4) If $a \leq b$, then $\mathcal{N}(A, b)(x) \leq \mathcal{N}(A, a)(x)$;

(5) $\mathcal{N}(A, a)(x) = \bigvee\{\mathcal{N}(B, a)(x) \mid B(y) \leq \mathcal{N}(A, a)(y) \text{ for all } y \in X\}$;

(6) If $\{A_i \mid i \in J\} \subseteq L^X$ is co-directed, then $\mathcal{N}\big(\bigwedge_{j\in J} A_j, a\big)(x) = \bigwedge_{j\in J} \mathcal{N}(A_j, a)(x)$;

(7) For $\{a_j : j \in J\} \subseteq M_{0_M}$, if $A(x) = \mathcal{N}(A, a_j)(x)$ for all $j \in J, x \in X$, then $A(x) = \mathcal{N}\big(A, \bigvee_{j\in J} a_j\big)(x)$.

If $\mathcal{N}$ is a concave (L, M)-fuzzy neighborhood operator on X, then the pair $(X, \mathcal{N})$ is called a concave (L, M)-fuzzy neighborhood space on X. Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be concave (L, M)-fuzzy neighborhood spaces, and let $g : X \to Y$ be a mapping. If for all $x \in X$, $A \in L^X$, and $a \in M_{0_M}$, $\mathcal{N}_Y(A, a)(g(x)) \leq \mathcal{N}_X\big(g^{\leftarrow}(A), a\big)(x)$, then g is called a concave (L, M)-fuzzy neighborhood-preserving function (abbreviated as (L, M)-NP). The category with concave (L, M)-fuzzy neighborhood spaces as objects and (L, M)-NPs as morphisms is denoted by (L, M)-**FN**.

First, we induce a concave (L, M)-fuzzy neighborhood operator via an (L, M)-fuzzy concave structure.

Theorem 3.1. *Let $(X, \mathcal{A})$ be an (L, M)-fuzzy concave space. Define a mapping $\mathcal{N}_{\mathcal{A}} : L^X \times M_{0_M} \to L^X$ as follows: for all $x \in X$, $A \in L^X$, and $a \in M_{0_M}$,*

$$\mathcal{N}_{\mathcal{A}}(A, a)(x) = \bigvee \{B(x) \in L \mid B \leq A,\ \mathcal{A}(B) \geq a\}, \tag{1}$$

then $\mathcal{N}_{\mathcal{A}}$ is a concave (L, M)-fuzzy neighborhood operator on X.

Proof. Only need to prove that $\mathcal{N}_{\mathcal{A}}$ satisfies (1)-(7) of Definition 3.1.

(1) For all $a \in M_{0_M}$, since $\mathcal{A}(1_X) \geq a$, we have $\mathcal{N}_{\mathcal{A}}(1_X, a)(x) = 1_L$.

(2) By the definition of $\mathcal{N}_{\mathcal{A}}$, for all $x \in X$, $\mathcal{N}_{\mathcal{A}}(A, a)(x) = \bigvee \{B(x) \in L \mid B \leq A,\ \mathcal{A}(B) \geq a\} \leq A(x)$.

(3) If $A \leq B$, then

$$\mathcal{N}_{\mathcal{A}}(A, a)(x) = \bigvee \{C(x) \in L \mid C \leq A,\ \mathcal{A}(C) \geq a\} \leq \bigvee \{C(x) \in L \mid C \leq B,\ \mathcal{A}(C) \geq a\} = \mathcal{N}_{\mathcal{A}}(B, a)(x).$$

(4) Let $a \leq b$. From (2) and (3), we know $\mathcal{N}_{\mathcal{A}}\big(\mathcal{N}_{\mathcal{A}}(A, b), a\big)(x) \leq \mathcal{N}_{\mathcal{A}}(A, b)(x)$ and $\mathcal{N}_{\mathcal{A}}\big(\mathcal{N}_{\mathcal{A}}(A, b), a\big)(x) \leq \mathcal{N}_{\mathcal{A}}(A, a)(x)$. Also, by the definition of $\mathcal{N}_{\mathcal{A}}$, for all $x \in X$,

$$\mathcal{N}_{\mathcal{A}}(A, a)(x) = \bigvee \{B(x) \in L \mid B \leq A,\ \mathcal{A}(B) \geq a\} = \Big[\bigvee \{B \in L \mid B \leq A,\ \mathcal{A}(B) \geq a\}\Big](x).$$

Thus, $\mathcal{A}\big(\mathcal{N}_{\mathcal{A}}(A, b)\big) = \mathcal{A}\big(\bigvee \{B \in L \mid B \leq A,\ \mathcal{A}(B) \geq b\}\big) \geq b \geq a$. This implies

$$\mathcal{N}_{\mathcal{A}}\big(\mathcal{N}_{\mathcal{A}}(A, b), a\big)(x) = \bigvee \{B(x) \in L \mid B \leq \mathcal{N}_{\mathcal{A}}(A, b),\ \mathcal{A}(B) \geq a\} \geq \mathcal{N}_{\mathcal{A}}(A, b)(x).$$

Hence, $\mathcal{N}_{\mathcal{A}}\big(\mathcal{N}_{\mathcal{A}}(A, b), a\big)(x) = \mathcal{N}_{\mathcal{A}}(A, b)(x)$, and therefore $\mathcal{N}_{\mathcal{A}}(A, b)(x) \leq \mathcal{N}_{\mathcal{A}}(A, a)(x)$.

(5) On the one hand, suppose for all $y \in X$, $B(y) \leq \mathcal{N}_{\mathcal{A}}(A,a)(y)$. Then $B \leq \mathcal{N}_{\mathcal{A}}(A,a)$, so $\mathcal{A}(\mathcal{N}_{\mathcal{A}}(A,a)) \geq a$, and $\mathcal{N}_{\mathcal{A}}(\mathcal{N}_{\mathcal{A}}(A,a),a)(x) = \bigvee\{B(x) \in L \mid B \leq \mathcal{N}_{\mathcal{A}}(A,a),\ \mathcal{A}(B) \geq a\} \geq \mathcal{N}_{\mathcal{A}}(A,a)(x)$. Thus, $\bigvee\{\mathcal{N}_{\mathcal{A}}(B,a)(x) \mid B(y) \leq \mathcal{N}_{\mathcal{A}}(A,a)(y)\ \forall y \in X\} \geq \mathcal{N}_{\mathcal{A}}(\mathcal{N}_{\mathcal{A}}(A,a),a)(x) \geq \mathcal{N}_{\mathcal{A}}(A,a)(x)$. On the other hand, if $B(y) \leq \mathcal{N}_{\mathcal{A}}(A,a)(y)\ \forall y \in X$, then by the definition of $\mathcal{N}_{\mathcal{A}}$, $B \leq A$. Hence, for all $x \in X$, $\mathcal{N}_{\mathcal{A}}(A,a)(x) \geq \mathcal{N}_{\mathcal{A}}(B,a)(x)$, so $\mathcal{N}_{\mathcal{A}}(A,a)(x) \geq \bigvee\{\mathcal{N}_{\mathcal{A}}(B,a)(x) \mid B(y) \leq \mathcal{N}_{\mathcal{A}}(A,a)(y)\ \forall y \in X\}$.

Combining the above, we conclude $\mathcal{N}_{\mathcal{A}}(A,a)(x) = \bigvee\{\mathcal{N}_{\mathcal{A}}(B,a)(x) \mid B(y) \leq \mathcal{N}_{\mathcal{A}}(A,a)(y)\ \forall y \in X\}$.

(6) Let $\{A_j\}_{j\in J} \overset{\text{codir}}{\subseteq} L^X$), Then $\{\mathcal{N}_{\mathcal{A}}(A_j,a) : j \in J\} \overset{\text{codir}}{\subseteq} L^X$. On the one hand, by (3), for all $j \in J$, $\mathcal{N}_{\mathcal{A}}(\bigwedge_{j\in J} A_j, a)(x) \leq \mathcal{N}_{\mathcal{A}}(A_j,a)(x)$. Hence,

$$\mathcal{N}_{\mathcal{A}}\big(\bigwedge_{j\in J} A_j, a\big)(x) \leq \bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A_j,a)(x).$$

On the other hand, by (2), $\bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A_j,a) \leq \bigwedge_{j\in J} A_j$. Also, from the definition of $\mathcal{A}$,

$$\mathcal{A}\big(\bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A_j,a)\big) \geq \bigwedge_{j\in J} \mathcal{A}\big(\mathcal{N}_{\mathcal{A}}(A_j,a)\big) \geq a.$$

Thus,

$$\mathcal{N}_{\mathcal{A}}\big(\bigwedge_{j\in J} A_j, a\big)(x) = \bigvee\left\{B(x) \in L \mid B \leq \bigwedge_{j\in J} A_j,\ \mathcal{A}(B) \geq a\right\} \geq \left[\bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A_j,a)\right](x) = \bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A_j,a)(x).$$

Combining the above, we have $\mathcal{N}_{\mathcal{A}}(\bigwedge_{j\in J} A_j, a)(x) = \bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A_j,a)(x)$.

(7) Let $\{a_j : j \in J\} \subseteq M_{0_M}$, and for all $j \in J$, $x \in X$, $\mathcal{N}_{\mathcal{A}}(A,a_j)(x) = A(x)$. Thus, $\mathcal{A}(A) = \mathcal{A}(\mathcal{N}_{\mathcal{A}}(A,a_j)) \geq a_j$, which further implies $\mathcal{A}(A) \geq \bigvee_{j\in J} a_j$. By (4), we have

$$A(x) = \bigwedge_{j\in J} \mathcal{N}_{\mathcal{A}}(A,a_j)(x) \geq \mathcal{N}_{\mathcal{A}}\big(A, \bigvee_{j\in J} a_j\big)(x) = \bigvee\left\{B(x) \in L \mid B \leq A,\ \mathcal{A}(B) \geq \bigvee_{j\in J} a_j\right\} \geq A(x).$$

Therefore, $\mathcal{N}_{\mathcal{A}}(A, \bigvee_{j\in J} a_j)(x) = A(x)$. □

Next, we induce an (L,M)-fuzzy concave structure via a concave (L,M)-fuzzy neighborhood operator.

Theorem 3.2. *Let $(X,\mathcal{N})$ be a concave (L,M)-fuzzy neighborhood space. Define a mapping $\mathcal{A}_{\mathcal{N}} : L^X \to M$ as follows: for all $A \in L^X$ and $a \in M_{0_M}$,*

$$\mathcal{A}_{\mathcal{N}}(A) = \bigvee\{a \in M_{0_M} \mid A = \mathcal{N}(A,a)\}. \tag{2}$$

Then $\mathcal{A}_{\mathcal{N}}$ is an (L,M)-fuzzy concave structure on X.

Proof. Only need to prove that $\mathcal{A}_{\mathcal{N}}$ satisfies (1)-(3) of Definition 2.14

(1) Since $1_M \in M_{0_M}$, by Definition 3.1 (1) and (2), we have $\mathcal{N}(1_X,1_M) = 1_X$, $\mathcal{N}(0_X,1_M) = 0_X$. Thus, $\mathcal{A}_{\mathcal{N}}(0_X) = \mathcal{A}_{\mathcal{N}}(1_X) = 1_M$.

(2) Let $b \in \beta^*\Big(\bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j)\Big)$. This implies $b \prec \bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j)$ and $b \in J(M)$. Since $\bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j) \leq \mathcal{A}_{\mathcal{N}}(A_j)$ for all $j \in J$, the order property of $\prec$ yields $b \prec \mathcal{A}_{\mathcal{N}}(A_j)$ for all $j \in J$. Therefore, there exists $a_j \in M_{0_M}$ such that $A_j = \mathcal{N}(A_j, a_j)$ and $b \prec a_j$. Let $a_0 = \bigwedge_{j\in J} a_j$, then $b \leq a_0$ and $a_0 \in M_{0_M}$. By Definition 3.1 (3) and (4), $\mathcal{N}\big(\bigvee_{j\in J} A_j, a_0\big) \geq \mathcal{N}\big(\bigvee_{j\in J} A_j, a_j\big) \geq \mathcal{N}(A_j, a_j)$. Thus, $\mathcal{N}\big(\bigvee_{j\in J} A_j, a_0\big) \geq \bigvee_{j\in J}\mathcal{N}(A_j, a_j) = \bigvee_{j\in J} A_j$.

On the other hand, by Definition 3.1 (2), $\mathcal{N}\big(\bigvee_{j\in J} A_j, a_0\big) \leq \bigvee_{j\in J} A_j$. Therefore,

$$\mathcal{N}\big(\bigvee_{j\in J} A_j, a_0\big) = \bigvee_{j\in J} A_j.$$

Hence, $\mathcal{A}_{\mathcal{N}}\big(\bigvee_{j\in J} A_j\big) \geq a_0 \geq b$, which implies $\mathcal{A}_{\mathcal{N}}\big(\bigvee_{j\in J} A_j\big) \geq \bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j)$.

(3) Let $\{A_j\}_{j\in J} \overset{\text{codir}}{\subseteq} L^X$, and let $b \in \beta^*\Big(\bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j)\Big)$. This implies $b \prec \bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j)$ and $b \in J(M)$. So, $b \prec \mathcal{A}_{\mathcal{N}}(A_j)$ for all $j \in J$. By the definition of $\mathcal{A}_{\mathcal{N}}$, for each $j \in J$, there exists $a_j \in M_{0_M}$ such that $A_j = \mathcal{N}(A_j, a_j)$ and $b \prec a_j$. Define $a_0 = \bigwedge_{j\in J} a_j$. Since $b \prec a_j$ for all j, the meet property of $\prec$ implies $b \leq a_0$, and clearly $a_0 \in M_{0_M}$. Applying Definition 3.1 (4) and (6), we derive

$$\bigwedge_{j\in J} A_j \geq \mathcal{N}\big(\bigwedge_{j\in J} A_j, a_0\big) = \bigwedge_{j\in J}\mathcal{N}(A_j, a_0) \geq \bigwedge_{j\in J}\mathcal{N}(A_j, a_j) = \bigwedge_{j\in J} A_j.$$

It follows that $\bigwedge_{j\in J} A_j = \mathcal{N}\big(\bigwedge_{j\in J} A_j, a_0\big)$, which implies $\mathcal{A}_{\mathcal{N}}\big(\bigwedge_{j\in J} A_j\big) \geq a_0 \geq b$. Hence $\mathcal{A}_{\mathcal{N}}\big(\bigwedge_{j\in J} A_j\big) \geq \bigwedge_{j\in J}\mathcal{A}_{\mathcal{N}}(A_j)$.

□

Proposition 3.1. *If $g : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ is an (L, M)-CAP, then $g : (X, \mathcal{N}_{\mathcal{A}_X}) \to (Y, \mathcal{N}_{\mathcal{A}_Y})$ is an (L, M)-NP.*

Proof. If $g : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ is an (L, M)-CAP, then $\forall A \in L^Y$, $\mathcal{A}_X\big(g^{\leftarrow}(A)\big) \geq \mathcal{A}_Y(A)$. Thus, for all $x \in X$, $A \in L^Y$, $a \in M_{0_M}$,

$$\begin{aligned}
\mathcal{N}_{\mathcal{A}_Y}(A, a)\big(g(x)\big) &= \bigvee\big\{B\big(g(x)\big) \in L \mid B \leq A,\ \mathcal{A}_Y(B) \geq a\big\} \\
&\leq \bigvee\big\{B\big(g(x)\big) \in L \mid g^{\leftarrow}(B) \leq g^{\leftarrow}(A),\ \mathcal{A}_X\big(g^{\leftarrow}(B)\big) \geq a\big\} \\
&= \bigvee\big\{g^{\leftarrow}(B)(x) \in L \mid g^{\leftarrow}(B) \leq g^{\leftarrow}(A),\ \mathcal{A}_X\big(g^{\leftarrow}(B)\big) \geq a\big\} \\
&\leq \bigvee\big\{C(x) \in L \mid C \leq g^{\leftarrow}(A),\ \mathcal{A}_X(C) \geq a\big\} = \mathcal{N}_{\mathcal{A}_X}\big(g^{\leftarrow}(A), a\big)(x).
\end{aligned}$$

Hence, $\mathcal{N}_{\mathcal{A}_Y}(A, a)\big(g(x)\big) \leq \mathcal{N}_{\mathcal{A}_X}\big(g^{\leftarrow}(A), a\big)(x)$. This indicates $g : (X, \mathcal{N}_{\mathcal{A}_X}) \to (Y, \mathcal{N}_{\mathcal{A}_Y})$ is an (L, M)-NP. □

Proposition 3.2. *If $g : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is an (L, M)-NP, then $g : (X, \mathcal{A}_{\mathcal{N}_X}) \to (Y, \mathcal{A}_{\mathcal{N}_Y})$ is an (L, M)-CAP.*

Proof. If $g : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is an (L, M)-NP, then for all $x \in X$, $A \in L^X$, $a \in M_{0_M}$,

$$\mathcal{N}_Y(A, a)(g(x)) \leq \mathcal{N}_X\big(g^{\leftarrow}(A), a\big)(x).$$

Therefore,

$$\begin{aligned}\mathcal{A}_{\mathcal{N}_Y}(A) &= \bigvee\{a\in M_{0_M}\mid A(y)=\mathcal{N}_Y(A,a)(x),\forall y\in Y\}\\ &\leq \bigvee\{a\in M_{0_M}\mid A(g(x))=\mathcal{N}_Y(A,a)(g(x))\}\\ &\leq \bigvee\big\{a\in M_{0_M}\mid g^{\leftarrow}(A)(x)=\mathcal{N}_X\big(g^{\leftarrow}(A),a\big)(x),\ \forall x\in X\big\}\\ &= \bigvee\big\{a\in M_{0_M}\mid g^{\leftarrow}(A)=\mathcal{N}_X\big(g^{\leftarrow}(A),a\big)\big\}=\mathcal{A}_{\mathcal{N}_X}\big(g^{\leftarrow}(A)\big).\end{aligned}$$

Hence $\mathcal{A}_{\mathcal{N}_Y}(A)\leq\mathcal{A}_{\mathcal{N}_X}\big(g^{\leftarrow}(A)\big)$. This shows that $g:(X,\mathcal{A}_{\mathcal{N}_X})\to(Y,\mathcal{A}_{\mathcal{N}_Y})$ is an (L,M)-CAP. □

From Theorems 3.1 and 3.2, Propositions 3.1 and 3.2, we obtain the following two functors:

$$\mathbb{A}_{\mathbb{N}}:\begin{cases}(L,M)\text{-}\mathbf{FN}\to(L,M)\text{-}\mathbf{FA},\\ (X,\mathcal{N})\mapsto(X,\mathcal{A}_{\mathcal{N}}),\\ f\mapsto f,\end{cases}\qquad \mathbb{N}_{\mathbb{A}}:\begin{cases}(L,M)\text{-}\mathbf{FA}\to(L,M)\text{-}\mathbf{FN},\\ (X,\mathcal{A})\mapsto(X,\mathcal{N}_{\mathcal{A}}),\\ f\mapsto f.\end{cases}$$

Finally, we will prove that $\mathbb{A}_{\mathbb{N}}$ and $\mathbb{N}_{\mathbb{A}}$ are isomorphic functors.

Theorem 3.3. *The categories (L,M)-**FN** and (L,M)-**FA** are isomorphic.*

Proof. It is necessary to prove that $\mathbb{A}_{\mathbb{N}}\circ\mathbb{N}_{\mathbb{A}}=\mathbb{I}_{(L,M)\text{-}\mathbf{FA}}$ and $\mathbb{N}_{\mathbb{A}}\circ\mathbb{A}_{\mathbb{N}}=\mathbb{I}_{(L,M)\text{-}\mathbf{FN}}$, that is, for any (L,M)-fuzzy concave space $(X,\mathcal{A})$ and concave (L,M)-fuzzy neighborhood space $(X,\mathcal{N})$, the following conditions are satisfied:

(1) $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}=\mathcal{A}$;

(2) $\mathcal{N}_{\mathcal{A}_{\mathcal{N}}}=\mathcal{N}$.

For (**1**), let $b\in\beta^*(\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}(A))$, then $b\in J(M)$ and $b\prec\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}(A)=\bigvee\{a\in M_{0_M}\mid A=\mathcal{N}_{\mathcal{A}}(A,a)\}$. Thus, there exists $a_0\in M_{0_M}$ such that $A=\mathcal{N}_{\mathcal{A}}(A,a_0)$ and $b\leq a_0$. Therefore, $\mathcal{A}(A)=\mathcal{A}(\mathcal{N}_{\mathcal{A}}(A,a_0))\geq b$. Hence, $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}(A)\leq\mathcal{A}(A)$. Conversely, when $\mathcal{A}(A)=0_M$, it is easy to see that $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}(A)\geq 0_M=\mathcal{A}(A)$. If $\mathcal{A}(A)\in M_{0_M}$, then from formula (1), we have $A(x)\geq\mathcal{N}_{\mathcal{A}}(A,\mathcal{A}(A))(x)=\bigvee\{B(x)\in L\mid B\leq A,\ \mathcal{A}(B)\geq\mathcal{A}(A)\}\geq A(x)$. Thus, $A=\mathcal{N}_{\mathcal{A}}(A,\mathcal{A}(A))$. From the definition of $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}$, we know that $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}(A)\geq\mathcal{A}(A)$. The above proof shows that $\mathcal{A}_{\mathcal{N}_{\mathcal{A}}}=\mathcal{A}$.

For (**2**), let $B\leq A$ and $\mathcal{A}_{\mathcal{N}}(B)\geq a$. Then from Definition 3.1 (2), (4),(7) and formula (2), for all $x\in X$, $B(x)\geq\mathcal{N}(B,a)(x)\geq\mathcal{N}(B,\mathcal{A}_{\mathcal{N}}(B))(x)=\mathcal{N}(B,\bigvee\{a\in M_{0_M}\mid B=\mathcal{N}(B,a)\})(x)=B(x)$. Thus, $B(x)=\mathcal{N}(B,a)(x)\leq\mathcal{N}(A,a)(x)$. Therefore, $\mathcal{N}_{\mathcal{A}_{\mathcal{N}}}(A,a)=\bigvee\big\{B\in L^X\mid B\leq A,\ \mathcal{A}_{\mathcal{N}}(B)\geq a\big\}\leq\mathcal{N}(A,a)$.

Conversely, from Definition 3.1 (2) and (5), we have

$$\mathcal{N}\big(\mathcal{N}(A,a),a\big)(x)\leq\mathcal{N}(A,a)(x)=\bigvee\{\mathcal{N}(B,a)(x)\mid B(y)\leq\mathcal{N}(A,a)(y),\ \forall y\in X\}\leq\mathcal{N}\big(\mathcal{N}(A,a),a\big)(x).$$

Therefore,

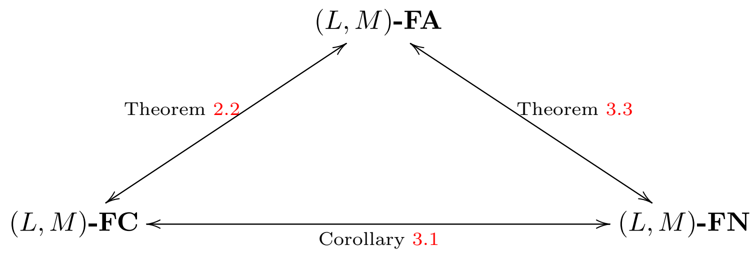
$$\mathcal{N}_{\mathcal{A}_{\mathcal{N}}}(A,a)=\bigvee\big\{B\in L^X\mid B\leq A,\ \mathcal{A}_{\mathcal{N}}(B)\geq a\big\}\geq\bigvee\big\{B\in L^X\mid A\geq B=\mathcal{N}(B,a)\big\}\geq\mathcal{N}(A,a).$$

The above proof shows that $\mathcal{N}_{\mathcal{A}_{\mathcal{N}}}=\mathcal{N}$. □

Corollary 3.1. *The categories (L,M)-**FC** and (L,M)-**FN** are isomorphic (As shown in Figure 1), established by mutually inverse functors that preserve algebraic structures. Specifically, the isomorphisms are realized via the identities $\mathcal{N}_{\mathcal{C}} = \mathcal{N}_{\mathcal{A}_{\mathcal{C}}}$ and $\mathcal{C}_{\mathcal{N}} = \mathcal{C}_{\mathcal{A}_{\mathcal{N}}}$, which induce the following chains of equalities:*

$$\mathcal{C}_{\mathcal{N}_{\mathcal{C}}} = \mathcal{C}_{\mathcal{N}_{\mathcal{A}_{\mathcal{C}}}} = \mathcal{C}_{\mathcal{A}_{\mathcal{N}_{\mathcal{A}_{\mathcal{C}}}}} = \mathcal{C}_{\mathcal{A}_{\mathcal{C}}} = \mathcal{C}; \quad \mathcal{N}_{\mathcal{C}_{\mathcal{N}}} = \mathcal{N}_{\mathcal{C}_{\mathcal{A}_{\mathcal{N}}}} = \mathcal{N}_{\mathcal{A}_{\mathcal{C}_{\mathcal{A}_{\mathcal{N}}}}} = \mathcal{N}_{\mathcal{A}_{\mathcal{N}}} = \mathcal{N}.$$

Figure 1. Category isomorphisms among three categories



4. S_3 and S_4 separation axioms in (L,M)-fuzzy convex space

In this section, we define the S_3 and S_4 separation axioms in (L,M)-fuzzifying convex spaces and analyze their relationships with S_1 and S_2. In particular, we investigate the hereditary property of the S_3 axiom in subspaces and its preservation under product spaces.

Definition 4.1. Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space.

(**1**) The S_3 separation axiom for $(X,\mathcal{C})$ is given by

$$S_3(X,\mathcal{C}) = \bigwedge_{A\in L^X} \bigwedge_{\substack{a\in J(L^X)\\ a\not\leq A}} \left(\mathcal{C}(A) \to \bigvee_{\substack{B\in L^X\\ a\not\leq B\geq A}} \mathcal{H}_{\mathcal{C}}(B) \right).$$

(**2**) The S_4 separation axiom for $(X,\mathcal{C})$ is given by

$$S_4(X,\mathcal{C}) = \bigwedge_{\substack{A,B\in L^X\\ A\leq B}} \left(\mathcal{C}(A) \wedge \mathcal{C}(B') \to \bigvee_{\substack{C\in L^X\\ B\geq C\geq A}} \mathcal{H}_{\mathcal{C}}(C) \right).$$

The following theorem establishes that the S_3 separation axiom in an (L,M)-fuzzy convex space possess the hereditary property.

Theorem 4.1. *Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space, and let $(Y,\mathcal{C}|_Y)$ be a subspace of $(X,\mathcal{C})$. Then:*

$$S_3(X,\mathcal{C}) \leq S_3(Y,\mathcal{C}|_Y).$$

Proof. Take any $\alpha \in M$ satisfying $\alpha \prec \bigwedge_{A\in L^X} \bigwedge_{\substack{a\not\leq A\\ a\in J(L^X)}} \left(\mathcal{C}(A) \to \bigvee_{a\not\leq B\geq A} \mathcal{H}_{\mathcal{C}}(B) \right)$. For each $A \in L^X$ and $a \in J(L^X)$ with $a \not\leq A$, by the definition of $\prec$ in M, we have $\alpha \leq \mathcal{C}(A) \to \bigvee_{a\not\leq B\geq A} \mathcal{H}_{\mathcal{C}}(B)$. By Lemma 2.1, this implies

$$\alpha \wedge \mathcal{C}(A) \leq \bigvee_{a \not\leq B \geq A} \mathcal{H}_{\mathcal{C}}(B) = \bigvee_{a \not\leq B \geq A} \big(\mathcal{C}(B) \wedge \mathcal{C}(B')\big). \tag{3}$$

Now, consider arbitrary $D \in L^Y$ and $b \in J(L^Y)$ with $b \not\leq D$. Let $V \in L^X$ be any extension of D (i.e., $V|_Y = D$; such extensions exist, e.g., the minimal extension with $V(x) = \mathbf{0}$ for $x \notin Y$). Since $b \in J(L^Y)$ (join-irreducible elements in L^Y), it is a fuzzy point: $b = y_\lambda$ with support $y \in Y$ and height $\lambda \in J(L)$ (join-irreducible in L). Define its canonical extension to X as $b^* \in L^X$, where

$$b^*(x) = \begin{cases} \lambda, & x = y \in Y, \\ \mathbf{0}, & x \in X \setminus \{y\}. \end{cases}$$

Clearly, $b^* \in J(L^X)$.

We claim $b^* \not\leq V$. Suppose for contradiction that $b^* \leq V$. Then $b^*(y) \leq V(y)$, so $\lambda \leq V(y)$. But $V|_Y = D$ implies $V(y) = D(y)$, so $\lambda \leq D(y)$, which means $b = y_\lambda \leq D$. This contradicts $b \not\leq D$, so $b^* \not\leq V$. Applying inequality (3) with $A = V$ and $a = b^*$, we get

$$\alpha \wedge \mathcal{C}(V) \leq \bigvee_{b^* \not\leq B \geq V} \big(\mathcal{C}(B) \wedge \mathcal{C}(B')\big). \tag{4}$$

For each B in the supremum above, consider its restriction $E = B|_Y \in L^Y$. We verify:

- $E \geq D$: Since $B \geq V$, restricting to Y gives $E = B|_Y \geq V|_Y = D$.
- $b \not\leq E$: Since $b^* \not\leq B$, we have $b^*(y) \not\leq B(y)$, so $\lambda \not\leq B(y) = E(y)$, which implies $b = y_\lambda \not\leq E$.
- $\mathcal{H}_{\mathcal{C}}(B) \leq \mathcal{H}_{\mathcal{C}|_Y}(E)$: By definition of the restricted convexity operator, $\mathcal{C}(B) \leq \mathcal{C}|_Y(E)$ (since B extends E). And, $(B')|_Y = (B|_Y)' = E'$, so $\mathcal{C}(B') \leq \mathcal{C}|_Y(E')$. Thus, $\mathcal{C}(B) \wedge \mathcal{C}(B') \leq \mathcal{C}|_Y(E) \wedge \mathcal{C}|_Y(E') = \mathcal{H}_{\mathcal{C}|_Y}(E)$.

By (4), we have

$$\alpha \wedge \mathcal{C}(V) \leq \bigvee_{b \not\leq E \geq D} \mathcal{H}_{\mathcal{C}|_Y}(E). \tag{5}$$

Note that $\mathcal{C}|_Y(D) = \bigvee_{V|_Y = D} \mathcal{C}(V)$. Using the distributivity of $\wedge$ over $\bigvee$ in complete lattices:

$$\alpha \wedge \mathcal{C}|_Y(D) = \alpha \wedge \left(\bigvee_{V|_Y = D} \mathcal{C}(V) \right) = \bigvee_{V|_Y = D} \big(\alpha \wedge \mathcal{C}(V)\big).$$

By (5), each term $\alpha \wedge \mathcal{C}(V)$ is bounded above by $\bigvee_{b \not\leq E \geq D} \mathcal{H}_{\mathcal{C}|_Y}(E)$, which is independent of V. Thus,

$$\alpha \wedge \mathcal{C}|_Y(D) \leq \bigvee_{b \not\leq E \geq D} \mathcal{H}_{\mathcal{C}|_Y}(E).$$

By Lemma 2.1, this implies $\alpha \leq \mathcal{C}|_Y(D) \to \bigvee_{b \not\leq E \geq D} \mathcal{H}_{\mathcal{C}|_Y}(E)$. Since $D \in L^Y$ and $b \in J(L^Y)$ with $b \not\leq D$ are arbitrary, we conclude

$$\alpha \leq \bigwedge_{D \in L^Y} \bigwedge_{\substack{b \not\leq D \\ b \in J(L^Y)}} \Big(\mathcal{C}|_Y(D) \to \bigvee_{b \not\leq E \geq D} \mathcal{H}_{\mathcal{C}|_Y}(E)\Big) = S_3(Y, \mathcal{C}|_Y).$$

Taking the supremum over all $\alpha \in M$ with $\alpha \prec S_3(X,\mathcal{C})$, we obtain $S_3(X,\mathcal{C}) \leq S_3(Y,\mathcal{C}|_Y)$. □

The following theorem provides the interconnections among the S_1, S_2, S_3 and S_4 separation axioms within an (L,M)-fuzzy convex space.

Theorem 4.2. *Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space. Then:*

(1) $S_3(X,\mathcal{C}) \wedge S_1(X,\mathcal{C}) \leq S_2(X,\mathcal{C})$*;*

(2) $S_1(X,\mathcal{C}) \wedge S_4(X,\mathcal{C}) \leq S_3(X,\mathcal{C})$.

Proof. (1) Take $\alpha \in M$ satisfying $\alpha \prec S_3(X,\mathcal{C}) \wedge S_1(X,\mathcal{C})$. Then

$$\alpha \prec S_3(X,\mathcal{C}) = \bigwedge_{A \in L^X} \bigwedge_{\substack{a \not\leq A \\ a \in J(L^X)}} \Big(\mathcal{C}(A) \to \bigvee_{a \not\leq B \geq A} \mathcal{H}_{\mathcal{C}}(B)\Big),$$

and $\alpha \prec S_1(X,\mathcal{C}) = \bigwedge_{a \in J(L^X)} \mathcal{C}(a)$.

Step 1: From $\alpha \prec S_3(X,\mathcal{C})$, for each $A \in L^X$ and $a \in J(L^X)$ with $a \not\leq A$, by Lemma 2.1, we have $\alpha \leq \mathcal{C}(A) \to \bigvee_{a \not\leq B \geq A} \mathcal{H}_{\mathcal{C}}(B)$, which implies that $\alpha \wedge \mathcal{C}(A) \leq \bigvee_{a \not\leq B \geq A} \mathcal{H}_{\mathcal{C}}(B)$.

Step 2: From $\alpha \prec S_1(X,\mathcal{C})$, for each $a \in J(L^X)$, we have $\alpha \leq \mathcal{C}(a)$. Consider arbitrary $a_1, a_2 \in J(L^X)$ with $a_1 \not\leq a_2$. Since $\alpha \prec S_1(X,\mathcal{C})$, by Step 2 we have $\alpha \leq \mathcal{C}(a_1)$, hence $\alpha = \alpha \wedge \mathcal{C}(a_1)$. Applying Step 1 with $A = a_1$ and $a = a_2$ (where $a_2 \not\leq a_1$ holds by assumption), we derive

$$\alpha = \alpha \wedge \mathcal{C}(a_1) \leq \bigvee_{a_2 \not\leq B \geq a_1} \mathcal{H}_{\mathcal{C}}(B).$$

Since a_1, a_2 are arbitrary with $a_1 \not\leq a_2$, taking the infimum over all such pairs yields

$$\alpha \leq \bigwedge_{\substack{a_1, a_2 \in J(L^X) \\ a_1 \not\leq a_2}} \bigvee_{a_2 \not\leq B \geq a_1} \mathcal{H}_{\mathcal{C}}(B) = S_2(X,\mathcal{C}).$$

As $\alpha \in M$ with $\alpha \prec S_3(X,\mathcal{C}) \wedge S_1(X,\mathcal{C})$ is arbitrary, we conclude $S_3(X,\mathcal{C}) \wedge S_1(X,\mathcal{C}) \leq S_2(X,\mathcal{C})$.

(2) Let $\alpha \in M$ be arbitrary with $\alpha \prec S_1(X,\mathcal{C}) \wedge S_4(X,\mathcal{C})$. By the property of the wedge-below relation in completely distributive lattices, this implies $\alpha \prec S_1(X,\mathcal{C})$ and $\alpha \prec S_4(X,\mathcal{C})$. By the definition of $S_1(X,\mathcal{C})$, for all $a \in J(L^X)$, $\alpha \prec \mathcal{C}(a)$. For $S_4(X,\mathcal{C})$, note that

$$S_4(X,\mathcal{C}) = \bigwedge_{\substack{A,B \in L^X \\ A \leq B}} \left(\mathcal{C}(A) \wedge \mathcal{C}(B') \to \bigvee_{\substack{C \in L^X \\ B \geq C \geq A}} \mathcal{H}_{\mathcal{C}}(C)\right).$$

Since $\alpha \prec S_4(X,\mathcal{C})$, for all $A, B \in L^X$ with $A \leq B$, $\alpha \leq \mathcal{C}(A) \wedge \mathcal{C}(B') \to \bigvee_{\substack{C \in L^X \\ B \geq C \geq A}} \mathcal{H}_{\mathcal{C}}(C)$.

By the adjunction property of lattice implication in completely distributive lattices, this implies:

$$\alpha \wedge \mathcal{C}(A) \wedge \mathcal{C}(B') \leq \bigvee_{\substack{C \in L^X \\ B \geq C \geq A}} \mathcal{H}_{\mathcal{C}}(C) \tag{6}$$

for all $A \leq B$.

Next, we prove that for all $A \in L^X$ and $a \in J(L^X)$ with $a \not\leq A$, the following holds:

$$\alpha \wedge \mathcal{C}(A) \leq \bigvee_{\substack{B \in L^X \\ a \not\leq B \geq A}} \mathcal{H}_{\mathcal{C}}(B).$$

To this end, for all $a \not\leq A$, take $b \in J(L^X)$ such that $b \leq A'$ and $b \not\leq a'$. By the order-reversing property of involution. Since $A \leq b'$, we may apply (6) with $B = b'$, yielding: $\alpha \wedge \mathcal{C}(A) \wedge \mathcal{C}(b) \leq \bigvee_{\substack{C \in L^X \\ b' \geq C \geq A}} \mathcal{H}_{\mathcal{C}}(C)$.

From $\alpha \prec S_1(X, \mathcal{C}) \leq \mathcal{C}(b)$ (since $b \in J(L^X)$), we have $\alpha \leq \mathcal{C}(b)$, so $\alpha \wedge \mathcal{C}(A) \wedge \mathcal{C}(b) = \alpha \wedge \mathcal{C}(A)$. Thus,

$$\alpha \wedge \mathcal{C}(A) \leq \bigvee_{\substack{C \in L^X \\ b' \geq C \geq A}} \mathcal{H}_{\mathcal{C}}(C).$$

We now show $\bigvee_{\substack{C \in L^X \\ b' \geq C \geq A}} \mathcal{H}_{\mathcal{C}}(C) \leq \bigvee_{\substack{C \in L^X \\ a \not\leq C \geq A}} \mathcal{H}_{\mathcal{C}}(C)$. For any $C \in L^X$ with $b' \geq C \geq A$, note $b \not\leq a' \implies a \not\leq b'$. Since $C \leq b'$, it follows that $a \not\leq C$ (for if $a \leq C$, then $a \leq b'$, contradicting $a \not\leq b'$). Thus C satisfies $a \not\leq C \geq A$, so the inclusion of joins holds.

Combining these results gives: $\alpha \wedge \mathcal{C}(A) \leq \bigvee_{\substack{B \in L^X \\ a \not\leq B \geq A}} \mathcal{H}_{\mathcal{C}}(B)$. By the definition of lattice implication, this is equivalent to: $\alpha \leq \mathcal{C}(A) \to \bigvee_{\substack{B \in L^X \\ a \not\leq B \geq A}} \mathcal{H}_{\mathcal{C}}(B)$. Taking the infimum over all $A \in L^X$ and $a \in J(L^X)$ with $a \not\leq A$, we conclude $\alpha \leq S_3(X, \mathcal{C})$. Since $\alpha \prec S_1(X, \mathcal{C}) \wedge S_4(X, \mathcal{C})$ was arbitrary, and in completely distributive lattices every element is the join of its wedge-below elements, it follows that $S_1(X, \mathcal{C}) \wedge S_4(X, \mathcal{C}) \leq S_3(X, \mathcal{C})$. □

At the end of this section, we prove that the S_3 separation axiom is preserved in product spaces under specific conditions. It is known that an M-fuzzifying Alexandrov topology is necessarily an M-fuzzifying convex structure. Thus, we formulate the following theorem:

Theorem 4.3. *Let $(X, \mathcal{C})$ be the product of M-fuzzifying convex spaces $\{(X_i, \mathcal{C}_i)\}_{i \in I}$. Suppose that $\mathcal{C}$ is furthermore an M-fuzzifying Alexandrov topology. Then $\bigwedge_{i \in I} S_3(X_j, \mathcal{C}_i) \leq S_3(X, \mathcal{C})$.*

Proof. Take any $\alpha \in M$ with $\alpha \prec \bigwedge_{i \in I} S_3(X_i, \mathcal{C}_i)$. By the wedge below property in completely distributive lattices, it follows that $\alpha \prec S_3(X_i, \mathcal{C}_i)$ for all $i \in I$. Recall from Definition 2.12 that $S_3(X_i, \mathcal{C}_i)$ is the infimum over $A_i \in 2^{X_i}$ and $x_i \notin A_i$ of the expression: $\mathcal{C}_i(A_i) \to \bigvee_{x_i \notin B_i \supseteq A_i} \mathcal{H}_{\mathcal{C}_i}(B_i)$. By Lemma 2.1(1), for each $i \in I$, $A_i \in 2^{X_i}$, and $x_i \notin A_i$, we have:

$$\alpha \wedge \mathcal{C}_i(A_i) \leq \bigvee_{x_i \notin B_i \supseteq A_i} \mathcal{H}_{\mathcal{C}_i}(B_i). \tag{7}$$

For all $A \in 2^X$ and $x \notin A$, We aim to show the following inequality: $\alpha \wedge \mathcal{C}(A) \leq \bigvee_{x \notin C \supseteq A} \mathcal{H}_{\mathcal{C}}(C)$. To this end, take $\gamma \in M$ such that $\gamma \prec \alpha \wedge \mathcal{C}(A)$, implying $\gamma \prec \alpha$ and $\gamma \prec \mathcal{C}(A)$. By Definitions 2.5 and 2.6, we have $\mathcal{C}(A) = \bigvee_{\bigcup^d_{j \in J} C_j = A} \bigwedge_{j \in J} \mathcal{B}_{\mathcal{G}}(C_j)$. Since $\gamma \prec \mathcal{C}(A)$, there exists a directed family $\{C_j\}_{j \in J}$ with $\bigcup^d_{j \in J} C_j = A$ and $\gamma \prec \mathcal{B}_{\mathcal{G}}(C_j)$ for all $j \in J$. By the definition of $\mathcal{B}_{\mathcal{G}}$, for each $j \in J$, there exists $\{D_t\}_{t \in T_j}$ with $\bigcap_{t \in T_j} D_t = C_j$ and $\gamma \prec \mathcal{G}(D_t)$ for all $t \in T_j$. By Definition 2.8: $\mathcal{G}(D_t) = \bigvee_{i \in I} \bigvee_{p_i^{-1}(B) = D_t} \mathcal{C}_i(B)$.

Thus, for each D_t, there exist $i_t \in I$ and $B_{i_t} \in 2^{X_{i_t}}$ such that $p_{i_t}^{-1}(B_{i_t}) = D_t$ and $\gamma \prec \mathcal{C}_{i_t}(B_{i_t})$. Since $x \notin A = \bigcup_{j\in J}^{d} C_j$, we have $x \notin C_j$ for all $j \in J$. For each $j \in J$, as $C_j = \bigcap_{t\in T_j} D_t$, there exists $t_j \in T_j$ with $x \notin D_{t_j}$. For this t_j, there exist $i_{t_j} \in I$ and $B_{i_{t_j}} \in 2^{X_{i_{t_j}}}$ such that $D_{t_j} = p_{i_{t_j}}^{-1}(B_{i_{t_j}})$ and $\gamma \prec \mathcal{C}_{i_{t_j}}(B_{i_{t_j}})$. Since $x \notin D_{t_j}$, we derive $x_{i_{t_j}} = p_{i_{t_j}}(x) \notin B_{i_{t_j}}$. Given $\gamma \prec \alpha$ and $\gamma \prec \mathcal{C}_{i_{t_j}}(B_{i_{t_j}})$, lattice properties give $\gamma \leq \alpha \wedge \mathcal{C}_{i_{t_j}}(B_{i_{t_j}})$. Take $\beta \in M$ with $\beta \prec \gamma$, then $\beta \prec \alpha \wedge \mathcal{C}_{i_{t_j}}(B_{i_{t_j}})$. By (7), $\beta \prec \bigvee_{x_{i_{t_j}} \notin E \supseteq B_{i_{t_j}}} \mathcal{H}_{\mathcal{C}_{i_{t_j}}}(E)$, and by the wedge below property, there exists $E \in 2^{X_{i_{t_j}}}$ with $x_{i_{t_j}} \notin E \supseteq B_{i_{t_j}}$ and $\beta \prec \mathcal{H}_{\mathcal{C}_{i_{t_j}}}(E)$. Define $\mathcal{D} = \bigcup_{j\in J} \left\{ p_{i_{t_j}}^{-1}(E) \mid E \subseteq X_{i_{t_j}},\ x_{i_{t_j}} \notin E \supseteq B_{i_{t_j}},\ \beta \prec \mathcal{H}_{\mathcal{C}_{i_{t_j}}}(E) \right\}$, and for each $j \in J$, let $D_j = \bigcap \{D \in \mathcal{D} \mid D \supseteq C_j\}$.

Step 1: $\{D_j\}_{j\in J}$ is directed. Since $\{C_j\}_{j\in J}$ is directed, for any $j_1, j_2 \in J$, there exists $j_3 \in J$ with $C_{j_1}, C_{j_2} \subseteq C_{j_3}$. Let $\mathcal{F}_3 = \{D \in \mathcal{D} \mid D \supseteq C_{j_3}\}$, $\mathcal{F}_1 = \{D \in \mathcal{D} \mid D \supseteq C_{j_1}\}$. As $C_{j_1} \subseteq C_{j_3}$, we have $\mathcal{F}_3 \subseteq \mathcal{F}_1$, so $\bigcap \mathcal{F}_1 \subseteq \bigcap \mathcal{F}_3$ (intersections reverse set inclusion). Thus, $D_{j_1} = \bigcap \mathcal{F}_1 \subseteq \bigcap \mathcal{F}_3 = D_{j_3}$. Similarly, $D_{j_2} \subseteq D_{j_3}$, so $\{D_j\}$ is directed.

Step 2: $x \notin \bigcup_{j\in J}^{d} D_j \supseteq A$. For $\bigcup_{j\in J}^{d} D_j \supseteq A$: By $D_j \supseteq C_j$, we have $\bigcup_{j\in J}^{d} D_j \supseteq \bigcup_{j\in J}^{d} C_j = A$. For $x \notin \bigcup_{j\in J}^{d} D_j$: For any $j \in J$, $D_j = \bigcap\{D \in \mathcal{D} \mid D \supseteq C_j\}$. For each $D \in \mathcal{D}$ with $D \supseteq C_j$, $D = p_{i_{t_k}}^{-1}(E)$ (for some $k \in J$) and $x_{i_{t_k}} \notin E$, so $x \notin D$. Thus, $x \notin \bigcap\{D \supseteq C_j\} = D_j$, implying $x \notin \bigcup_{j\in J}^{d} D_j$.

Step 3: $\mathcal{H}_{\mathcal{C}}(\bigcup_{j\in J}^{d} D_j) \geq \beta$. Each $D \in \mathcal{D}$ satisfies $D = p_i^{-1}(E)$ $(i \in I)$ and $\beta \prec \mathcal{C}_i(E) \wedge \mathcal{C}_i(X_i \setminus E)$. Since p_i is an (L, M)-CP function (Definition 2.8): $\mathcal{C}(D) = \mathcal{C}(p_i^{-1}(E)) \geq \mathcal{C}_i(E) \geq \beta$. For $D_j = \bigcap\{D \in \mathcal{D} \mid D \supseteq C_j\}$, by Definition 2.1 (2), $\mathcal{C}(D_j) \geq \bigwedge\{\mathcal{C}(D) \mid D \in \mathcal{D}, D \supseteq C_j\} \geq \beta$. Since $\{D_j\}$ is directed, Definition 2.2 gives: $\mathcal{C}\left(\bigcup_{j\in J}^{d} D_j\right) \geq \bigwedge_{j\in J} \mathcal{C}(D_j) \geq \beta$, and

$$\mathcal{C}(X \setminus \bigcup_{j\in J}^{d} D_j) = \mathcal{C}(\bigcap_{j\in J}(X \setminus D_j)) \geq \bigwedge_{j\in J} \mathcal{C}(X \setminus D_j).$$

For each $j \in J$, $X \setminus D_j = \bigcup\{X \setminus D \mid D \in \mathcal{D}, D \supseteq C_j\}$, where $D = p_i^{-1}(E)$ and $\beta \prec \mathcal{C}_i(X_i \setminus E)$ (by $\mathcal{H}_{\mathcal{C}_i}(E)$'s definition). Since p_i is an (L, M)-CP function, we have $\mathcal{C}(X \setminus D) = \mathcal{C}(p_i^{-1}(X_i \setminus E)) \geq \mathcal{C}_i(X_i \setminus E) \geq \beta$, and by $\mathcal{C}$ being an Alexandrov topology: $\mathcal{C}(X \setminus D_j) \geq \bigwedge \mathcal{C}(X \setminus D) \geq \beta$. Combining these: $\mathcal{H}_{\mathcal{C}}(\bigcup_{j\in J}^{d} D_j) = \mathcal{C}(\bigcup_{j\in J}^{d} D_j) \wedge \mathcal{C}(X \setminus \bigcup_{j\in J}^{d} D_j) \geq \beta \wedge \beta = \beta$. Thus, $\bigvee_{x\notin C\supseteq A} \mathcal{H}_{\mathcal{C}}(C) \geq \beta$ (since $\bigcup_{j\in J}^{d} D_j$ is a valid set C satisfying $x \notin C \supseteq A$). Given that $\beta \prec \gamma$ and $\gamma \prec \alpha \wedge \mathcal{C}(A)$ for arbitrary β, γ, we obtain $\alpha \wedge \mathcal{C}(A) \leq \bigvee_{x\notin C\supseteq A} \mathcal{H}_{\mathcal{C}}(C)$. Taking the infimum over all $A \in 2^X$ with $x \notin A$, and by the definition of $S_3(X, \mathcal{C})$, we obtain $\alpha \leq S_3(X, \mathcal{C})$. Since $\alpha \prec \bigwedge_{i\in I} S_3(X_i, \mathcal{C}_i)$ is arbitrary, we conclude: $\bigwedge_{i\in I} S_3(X_i, \mathcal{C}_i) \leq S_3(X, \mathcal{C})$. $\square$

The following example demonstrates that there exists a case satisfying the conditions of Theorem 4.3.

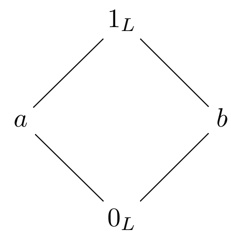
Example 4.1. *Let $Y = X = \{x\}$ be a singleton set, let $M = L = \{0_L, a, b, 1_L\}$ be a diamond-type lattice (As shown in Figure 2). Then $L^X = \{0_X, \underline{a}, \underline{b}, 1_X\}$ and $L^{X\times X} = \{0_{X\times X}, \underline{a}, \underline{b}, 1_{X\times X}\}$. Define two mappings*

$\mathcal{C}_1, \mathcal{C}_2 : L^X \longrightarrow M$ *as follows:*

$$\mathcal{C}_1(A) = \begin{cases} 1_L, & \text{if } A = 0_X, = 1_X, \\ b, & \text{if } A = \underline{a}, \underline{b}, \end{cases} \qquad \mathcal{C}_2(A) = \begin{cases} 1_L, & \text{if } A = 0_X, 1_X, \\ a, & \text{if } A = \underline{a}, \underline{b}. \end{cases}$$

Both $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *satisfy all axioms of* (L, M)*-fuzzy convex structures. For all* $A \in L^{X \times X}$*, we have* $(\mathcal{C}_1 \times \mathcal{C}_2)(A) = 1_L$*. Clearly,* $\mathcal{C}_1 \times \mathcal{C}_2$ *constitutes an* (L, M)*-fuzzy Alexandrov topology on* $X \times Y$*.*

Figure 2. Diamond Lattice $L = \{0_L, a, b, 1_L\}$



Remark 4.1. *In* (L, M)*-fuzzy convex spaces, the properties of* S_3 *and* S_4 *are analogous to regularity and normality in* (L, M)*-fuzzy topology (cf. [7]). It is well-known that in topology, regularity possesses the product property, while normality does not (e.g., the product of two lower limit topological spaces on* $\mathbb{R}$ *is not a normal space). Within the framework of* (L, M)*-fuzzy convex structures discussed in this section,* S_3 *does not necessarily satisfy the product property, which can be attributed to the following two aspects:*

(1) The degree to which set A *is a biconvex set,* $\mathcal{H}_{\mathcal{C}}(A)$*, namely,* A *is influenced by both* $\mathcal{C}(A)$ *and* $\mathcal{C}(A')$*.*

(2) The order relation $x_\lambda \leq \bigvee_{j \in J} B_j$ *does not generally imply the existence of some* $j \in J$ *such that* $x_\lambda \leq B_j$ *when* $\{B_j\}_{j \in J}$ *is a directed set. However, this implication holds when* J *is a finite set.*

5. Conclusion

This study systematically investigates concave (L, M)-fuzzy neighborhood operators and separation axioms in (L, M)-fuzzy convex spaces, with the key conclusions summarized as follows:

1. Regarding concave (L, M)-fuzzy neighborhood operators: This study proposes the formal definition of concave (L, M)-fuzzy neighborhood operators. Furthermore, by means of (L, M)-fuzzy concave structures, a bidirectional construction mechanism between concave (L, M)-fuzzy neighborhood operators and (L, M)-fuzzy convex structures is established. This mechanism provides a new method for characterizing the category of (L, M)-fuzzy convex spaces and enriches the theoretical tools for the analysis of (L, M)-fuzzy convex spaces.

2. Regarding separation axioms in (L, M)-fuzzy convex spaces: The S_3 and S_4 separation axioms are defined for (L, M)-fuzzy convex spaces. On this basis, this study explores the properties of the S_3 axiom and verifies its hereditary property in subspaces. It is also confirmed that the S_3 axiom is invariant under product operations, but this invariance is not universal and is restricted by specific constraints.

3. Specific constraints for the invariance of the S_3 axiom under product operations: Three aspects of conditions are clarified:

   (a) Product structure requirement: The product (L, M)-fuzzy convex structure $\prod_{j\in J} \mathcal{C}_j$ must further form an (L, M)-fuzzy Alexandrov topology (i.e., closed under arbitrary intersections);

   (b) Index set restriction: When the index set J is finite, the product property of the S_3 axiom always holds; when the index set is infinite, additional conditions are required to ensure the properties of directed unions.

4. Research limitation and theoretical implication: A notable limitation of this study is that the product property of the S_3 axiom does not hold unconditionally, and its application scope is restricted by the aforementioned conditions. This finding reveals an important difference in the properties of separation axioms between (L, M)-fuzzy convex spaces and classical topological spaces, providing a new perspective for understanding the structural characteristics of (L, M)-fuzzy convex spaces.

Fuzzy convex structures, especially Alexandrov fuzzy topology as their special subclass, serve as the core supporting tools of fuzzy rough set theory, and thus hold irreplaceable key value in both theoretical construction and practical application within the field of decision-making. From a theoretical perspective, fuzzy convex structures, by virtue of their essential convexity, establish a rigorous mathematical foundation for the set-based analysis of decision-making schemes, enabling precise clarification of the ordered correlations and optimization directions among various schemes. In contrast, Alexandrov fuzzy topology, as a special form of fuzzy convex structures, further relies on its accurate characterization of open and closed sets in fuzzy topological spaces to transform complex decision-making scenarios into measurable and derivable topological models, thereby effectively resolving the various challenges caused by information fuzziness and variable coupling during the decision-making process. In practical applications, relying on the basic framework of fuzzy convex structures and the distinctive advantages of Alexandrov fuzzy topology, the synergistic application of the two can provide precise analytical tools for key decision-making links such as risk assessment, path optimization and resource allocation. It helps decision-makers quickly lock in the optimal solution domain amid a wide range of alternative schemes, significantly enhancing the scientificity and effectiveness of decision-making. Future research will focus on the following direction: exploring applications of concave (L, M)-fuzzy neighborhood operators in fuzzy optimization and vector spaces.

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