Asia Mathematatika
Volume: 9 Issue: 3 (2025) Pages: 47 – 59
Available online at **www.asiamath.org**

Determination of the transitivity and primitivity of dihedral groups of prime degrees that are not p-groups using numerical approach

Ben O. Johnson*, Adagba T. Titus
Department of Mathematics, Federal University, Wukari.
Wukari, Nigeria.

Received: 12 Oct 2025 • **Accepted:** 15 Dec 2025 • **Published Online:** 31 Dec 2025

Abstract: In this study, we carried out further study on the transitive and primitive nature of dihedral group of prime degrees that are not p-groups by numerical approach. Transitivity and primitivity are two pivotal properties that provide deeper insights into group structures. Primitive groups represent the building blocks of all finite groups, akin to prime numbers in number theory. Transitivity, on the other hand, reflects a group's ability to act uniformly on a set, highlighting its symmetrical properties. A group G acting on a set Ω is said to be transitive on Ω if it has one orbit and so $\alpha^G = \Omega$ for all $\alpha \in \Omega$. Equivalently, G is transitive if for every pair of point $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$. A permutation group G acting on a non empty set Ω is called primitive if G acts transitively on Ω and G preserves no non trivial partition of Ω. In other words, a group G is said to be primitive on a set Ω if the only sets of imprimitivity are the trivial ones otherwise G is imprimitive on Ω. In this work we generated some dihedral groups of prime degrees that are not p -groups and used computational tools, including GAP (Groups, Algorithms, and Programming) coupled with maximality theorem to analyze their structures and action properties and discuss their transitive and primitive nature. The findings contribute to a deeper understanding of finite permutation groups, offering new insights into their classification and properties. This study not only enriches the theoretical framework of abstract algebra but also provides practical applications in areas such as cryptography, chemistry, and computational group theory.

Key words: transitive groups, primitive groups, prime degree, p-groups, numerical approach, group theory, GAP

1. Introduction

1.1. Background of the Study

Group theory plays a pivotal role in many areas of mathematics, especially where symmetry is a key consideration. Symmetry in any object is inherently tied to group theory, making it difficult to discuss symmetry without referencing this mathematical framework. As one of the foundational branches of abstract algebra, group theory seeks to classify groups up to isomorphism. This means that, for any given group, it should be possible to identify a corresponding known group via an isomorphism. The use of groups was first effectively applied in the early 19th century by mathematicians such as Joseph Louis Lagrange, Paolo Ruffini, and Évariste Galois. They utilized groups to understand how permutations of polynomial roots behaved (Galois and Singh, 1897 [12]). At that time, the concept of groups was not based on an axiomatic foundation. Subsequent developments by Augustin-Louis Cauchy and Arthur Cayley refined the theory, with Cayley (1854 [8]) proposing the first formal group postulates. However, these postulates were largely overlooked until Leopold Kronecker formalized the axioms for Abelian groups in 1870. Johanna Weber later provided definitions for finite and infinite groups

*Correspondence: bjobakpo@fuwukari.edu.ng

in 1882 and 1883, respectively (Kleiner, 1986 [16]). According to Cameron (2013 [7]), before 1850, the term "group" referred to a set G of transformations of a set Ω, such that G closed under composition, contained the identity transformation, and included the inverse of each element. This early interpretation corresponds to what we now refer to as a "permutation group." The requirement for every element of G to have an inverse implied that the function was both one-to-one and onto, effectively a permutation. In modern terms, a group is defined as a non-empty set G equipped with a binary operation, denoted *, which satisfies the associative property, contains an identity element, and ensures that every element has an inverse. Although this modern, axiomatic approach differs from earlier interpretations, the two
perspectives are essentially equivalent. Today, permutation groups are viewed as algebraic structures where the operation is function composition, which is inherently associative (Gallian, 2010 [11]; Roman, 2012 [18]).

Let Ω be a non-empty set. A permutation of Ω is a bijection $\alpha : \Omega \to \Omega$. The set of all permutations of Ω is denoted S_Ω. For a finite set $\Omega = \{1, 2, \ldots, n\}$, the set of permutations S_n, the symmetric group of degree n , with order $|S_n| = n!$. During his work, Joseph-Louis Lagrange observed permutation as arrangements, that is, as a list $i_1, i_2, \ldots, i_n$ with no repetition of any of the elements of Ω. Given an arrangement, $i_1, i_2, \ldots, i_n$, define a function $\alpha : \Omega \to \Omega$ by $\alpha(j) = i$ for all $j \in \Omega$. Thus, every rearrangement gives a bijection (Burness and Tong-Viet, 2016 [6]).

Dihedral groups, which describe the symmetries of regular polygons, provide important examples of finite permutation groups. These groups consist of rotations and reflections and have numerous applications in natural sciences and engineering (Cameron, 2013 [7]). For a regular polygon with n sides, dihedral groups are denoted as $D_{\rm n}$ or $D_{\rm 2n}$. In this work, the notation $D_{\rm n}$, representing the symmetry group of a polygon with n sides, is used.

When any group G, be it a dihedral or symmetric group acts on a set Ω, a typical point α is moved by elements of G to various other points. The set of these images is called the orbit of α under G, and we denote it by $\alpha^G := \{\alpha^g \mid g \in G\}$. A group G acting on a set Ω is said to be transitive on Ω if it has one orbit and so $\alpha^G = \Omega$ for all $\alpha \in \Omega$. Equivalently, G is transitive if for every pair of point $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$. A group which is not transitive is called intransitive, see Fawcett (2013 [10]). A permutation group G acting on a non empty set Ω is called primitive if G acts transitively on Ω and G preserves no non trivial partition of Ω. In other words, a group G is said to be primitive on a set Ω if the only sets of imprimitivity are the trivial ones otherwise G is imprimitive on Ω. (see Ben et al., 2022 [5] and Ben et al., [4]).

The concepts of transitivity and primitivity are fundamental for understanding group structures. Primitive groups serve as the building blocks of finite groups, akin to prime numbers in number theory, while transitivity highlights a group's ability to act uniformly on a set. (see Apine and Jelten, 2014 [1] and Apine et al., 2015 [2]). Graphical and numerical methods, combined with computational tools such as GAP, offer effective techniques for studying these algebraic structures (Hulpke et al., 2016 [15] and Neerajah and Subramanian, 2025 [17]). This study focuses on dihedral groups of prime degrees that are not p -groups, investigating their transitivity and primitivity using numerical approaches.

2. Materials and Method

In this work, knowledge of the basic facts from both the theory of abstract finite groups and the theory of permutation will be assumed throughout. Relevant theorems and results are given and quoted with example where necessary, in order to enhance proper understanding of the subject matter. We also use the Groups Algorithm and Programming (GAP) to enhance and validate our work.

2.1. Theorem (Thomas Judson, 2007)

The symmetric group on n letters, S_n, is a group with $n!$ elements, where the binary operation is the composition of maps.

Proof. The identity of S_n is just the identity map that sends 1 to 1,2 to $2, \ldots, n$ to n. If $f : S_n \rightarrow S_n$ is a permutation, then f^{-1} exists, since f is one-to-one and onto; hence, every permutation has an inverse. Composition of maps is associative, which makes the group operation associative. □

Lemma 2.1. *Let G be a dihedral group. Then $G = D_{\mathrm{n}}$ has 2 n distinct elements.*

Proof. Conventionally, we write $D_{\mathrm{n}} = \langle r, f \mid r^n = f^2 = 1, fr = r^{n-1}f = r^{-1}f \rangle$ and we say that D_{n} is the group generated by the elements r and f subject to the conditions

$$r^n = f^2 = 1; fr = r^{n-1}f = r^{-1}f \tag{1}$$

and the 2 n distinct elements of D_{n} are

$$1, r, r^2, \ldots, r^{n-1}, f, rf, r^2 f, \ldots, r^{n-1} f \tag{2}$$

Here r is a rotation about the centre of the polygon through angle $2\pi^c/n$ and f is a reflection about an axis of symmetry of the polygon. □

2.2. Group Action (Dixon and Mortimer, 1996 [9])

If a group G is acting on a subgroup H of G, then H is equipped with the restriction of the operation of G. Let G be a group and Ω be a non-empty set. We say that G acts on Ω (or that G permutes Ω) if to each $a \in \Omega$ and g_1, g_2 in G we have that $(ag_1)\,g_2 = a\,(g_1 g_2)$ and $ae = a$, where e is the identity element of G.

2.3. Transitivity

Let G be a permutation group on Ω, where Ω is a finite set.

1. We say that G is $\frac{1}{2}$ - transitive if all the orbits have the same size.

2. Suppose that G has just one orbit Ω. then for all $\mathrm{r} \in \Omega$, $r^G =$ Ω and as such for any $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$, and G is said to be transitive (or that G acts transitively) on Ω

3. The group G is said to be k -fold transitive (or, simply k -transitive) on Ω if, for any sequences $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_i \neq \alpha_j$ when $i \neq j; \beta_1, \beta_2, \ldots, \beta_k$ such that $\beta_1 \neq \beta_j$ when $i \neq j$ of k elements of Ω, there exists $g \in \mathrm{G}$ such that

$$\alpha_i^g = \beta_i \text{ for } 1 \leq i \leq k \tag{3}$$

Thus for $k = 2$ we have that for $\alpha_1, \alpha_2, \beta_1, \beta_2$ in Ω with $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ there exists $g \in G$ such that;

$$\alpha_1^g = \beta_1, \alpha_2^g = \beta_2. \tag{4}$$

and we say that G is doubly transitive.

Corollary 2.1 (Rotman, 1979 [20]). *A finite group G is a p-group if and only if $|G|$ is a power of p*

Proof. If $|G| = p^m$, then Langrage's theorem show that G is a p-group. Conversely, assume that there is a prime $q \neq p$ which divides $|G|$. By Cauchy's theorem, G contains an element of order q, and this contradicts G of being a p group. □

Lemma 2.2. *Let G be a dihedral group of any order, then G is transitive.*

Proof. For given α_i, α_j as any two vertices of the regular polygon with $i < j$, we readily see that $(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_n)^{j-i}$ is the rotation about the centre of the polygon through angle $2\pi^c/n$ (where n is the number of edges of the polygon) which take α_i to α_j. As such G is transitive. □

2.4. Primitivity (Araújo et al., 2016 [3])

A permutation group G acting on a non empty set Ω is said to be primitive on a set Ω if and only if it preserves the trivial block system otherwise G is imprimitive on Ω. For example, the group
$S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is primitive as $\{1, 2\}^{(123)} = \{2, 3\}$ implying that $\Delta^g \neq \Delta$ and $\Delta^g \cap \Delta \neq \varnothing$ for $\Delta = \{1, 2\}$.
On the other hand, a subset Δ of Ω is said to be a set of imprimitivity for the action of G on Ω, if for each $g \in G$, either $\Delta^g = \Delta$ or Δ^g and Δ are disjoint. In particular, Ω itself, the 1 -element subsets of Ω and the empty set are obviously sets of imprimitivity which are called trivial set of imprimitivity.
The group of symmetry $D_8 = \{(1), (1234), (13)(24), (1432), (13), (24), (12)(34), (14)(23)\}$ of the square with vertices $1, 2, 3, 4$ is imprimitive. For take $G_1 = \{(1), (24)\}$.
Let $H = \{(1), (13), (24), (13)(24)\}$ which is a normal subgroup of G. Then H is a group greater than G_1, but not equal to G.

Theorem 2.1 (Passman, 1968 [19]). *Let G be a non-trivial transitive permutation group on Ω. Then G is primitive iff $G_\alpha, (\alpha \in \Omega)$ is a maximal subgroup of G or equivalently, G is imprimitive if and only if there is a subgroup H of G properly lying between $G_\alpha, (\alpha \in \Omega)$ and G.*

Proof. Suppose G is imprimitive and ψ a non-trivial subset of imprimitivity of G. Let $H = \{g \in G \mid \psi^g = \psi\}$.
Clearly H is a subgroup of G and a proper subgroup of G because $\psi \subset \Omega$ and G is transitive.
Now choose $\alpha \in \psi$. If $g \in G$ then $\alpha^g = \alpha$, showing that $\alpha \in \psi \cap \psi^g$ and so $\psi = \psi^g$.
Hence $H \leq G$. Which follow that $G_\alpha \leq H \leq G$.
Since $|\psi| \neq 1$, choose $\beta \in \psi$ such that $\beta \neq \alpha$. By transitivity of G, there exist some $h \in G$ with $\alpha^h = \beta$ so that $h \in G_\alpha$. Now $\beta \in \psi \cap \psi^h$, so $\psi = \psi^h$ and $h \in H - G_\alpha$. Thus, $H \neq G_\alpha$.
Hence G_α is not a maximal subgroup.
Conversely, suppose that $G_\alpha < H < G$ for some subgroup H.
Let $\psi = \alpha^{\mathrm{H}}$. Since $H > G_\alpha, |\psi| \neq 1$.
Now if $\psi = \Omega$, then H is transitive on Ω and hence $|\Omega| = |G : G_\alpha| = |H : G_\alpha|$ showing that $H = G$, a contradiction.
Hence, $\psi = \Omega$.
Now we shall show that ψ is a subset of imprimitivity of G.
Let $g \in G$ and $\beta \in \psi \cap \psi^g$ then $\beta = \alpha^h = \alpha^{h'g}$ for some $h, h' \in H$.

Hence $\alpha^{h'gh^{-1}} = \alpha$. So $h'gh^{-1} \in G_\alpha < H$.
Thus $\psi = \psi^g$. Hence ψ is a non-trivial subset of imprimitivity. So G is imprimitive. □

3. Results and Discussion

3.1. Introduction

Here, we discuss in detail the transitivity and primitivity of dihedral groups of prime degrees that are not p-groups using numerical approach.

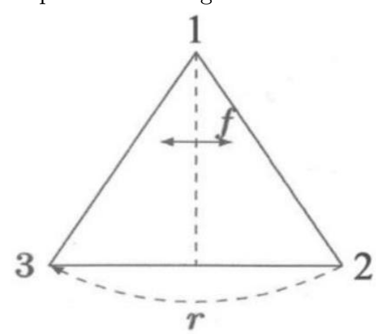
3.2. Transitivity and Primitivity of Dihedral Groups of Prime Degrees.

The following are the main results on the constructed Wreath Product group of degree n (where n is prime).

3.2.1. The Dihedral Group of Degree $\mathbf{n(n=3)}$

Consider a regular triangle T, with vertices labeled 1,2, and 3. We show T below, also using dotted lines to indicate a vertical line of symmetry of T and a rotation of T.

Figure 1. An Equilateral Triangle with vertices labeled 1, 2, 3.



Note that if we reflect T over the vertical dotted line (indicated in the picture by f), T maps onto itself, with 1 mapping to 1, and 2 and 3 mapping to each other. Similarly, if we rotate T clockwise by $120°$ (indicated in the picture by r), T again maps onto itself, this time with 1 mapping to 2,2 mapping to 3, and 3 mapping to 1. Both of these maps are called symmetries of $T; f$ is a reflection or flip, and r a rotation.

Of course, these are not the only symmetries of T. If we compose two symmetries of T, we obtain a symmetry of T : for instance, if we apply the map for to T (meaning first do r , then do f) we obtain reflection over the line connecting 2 to the midpoint of line segment 1-3. Similarly, if we apply the map fo (ror) to T (first do r twice, then do f) we obtain reflection over the line connecting 3 to the midpoint of line segment $1-2$. In fact, every symmetry of T can be obtained by composing applications of f and applications of r.

For convenience of notation, we omit the composition symbols, writing, for instance, fr for f o r, r o r as r^2, etc. It turns out there are exactly six symmetries of T, namely:

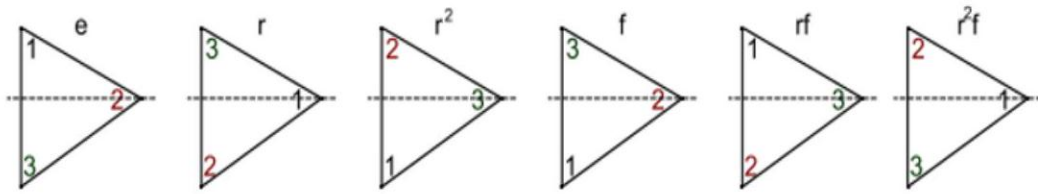
1. the map e from T to T sending every element to itself;
2. f (i.e, reflection over the line connecting 1 and the midpoint of 2-3);
3. r (that is, clockwise rotation by $120°$);

4. $\boldsymbol{r}^2$ (that is, clockwise rotation by 240°);

5. fr (i.e., reflection over the line connecting 2 and the midpoint of 1-3); and

6. fr^2 (i.e, reflection over the line connecting 3 and the midpoint of 1-2).

Figure 2. A labeled triangle after individual elements of D_3 have been applied



Clearly, $f^\circ = \mathrm{r}^\circ = \mathrm{e}$, the set
$D_3 = \{\boldsymbol{e}, f, \boldsymbol{r}, \boldsymbol{r}^2, fr, fr^2\}$ is the collection of all symmetries of T.
The Cayley table for the group D_3 is as follows.

Table 1. Table 1. Calay's table for D_3

$\times$	e	r	r^2	f	rf	r^2f
e	e	r	r^2	f	rf	r^2f
r	r	r^2	e	rf	r^2f	f
r^2	r^2	e	r	r^2f	f	rf
f	f	r^2f	rf	e	r^2	r
rf	rf	f	r^2f	r	e	r^2
r^2f	r^2f	rf	f	r^2	r	e

The elements of the group D_3 in cycle form is as follows.
$D_3 = \{(1), (2,3), (1,2), (1,2,3), (1,3,2), (1,3)\}$
Routing calculations shows that the stabilizers of the points 1,2, and 3 are respectively given by:
$G_1 = \{(1), (23)\}$
$G_2 = \{(1), (13)\}$
$G_3 = \{(1), (12)\}$
And the orbit of the points 1,2, and 3 is given by $1^G = 2^G = 3^G = \{1,2,3\}$
A group $G = D_3$ acting on a set $\Omega = \{1,2,3\}$ is said to be transitive on Ω if it has one orbit, and so $\alpha^{\mathrm{G}} = \Omega$ for all $\alpha \in \Omega$. A group which is not transitive is called intransitive. Thus, $D3 = \{(1), (1,2), (1,3), (2,3), (123), (132)\}$ is transitive.
$D_3 = \{(1), (12), (13), (23), (123), (132)\}$ is primitive as $\{1,2\}^{(123)} = \{2,3\}$ implying that $\Delta^g \neq \Delta$ and $\Delta^g \cap \Delta \neq \varnothing$ for $\Delta = \{1,2\}$.
On the other hand a subset Δ of Ω is said to be a set of imprimitivity for the action of G on Ω, if for each $g \in G$, either $\Delta^g = \Delta$ or Δ^g and Δ are disjoint. In particular, Ω itself, the 1-element subsets of Ω and the empty set are obviously sets of imprimitivity which are called trivial set of imprimitivity.

3.2.2. The Dihedral Group of Degree $\mathbf{n}(\mathbf{n}=\mathbf{5})$

Let's consider a pentagon with its corners numbered $1, 2, 3, 4$, and 5.

Figure 3. A labeled pentagon

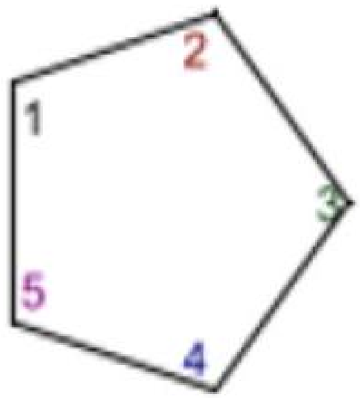


Figure 4. A labeled pentagon after individual elements of D_5 have been applied

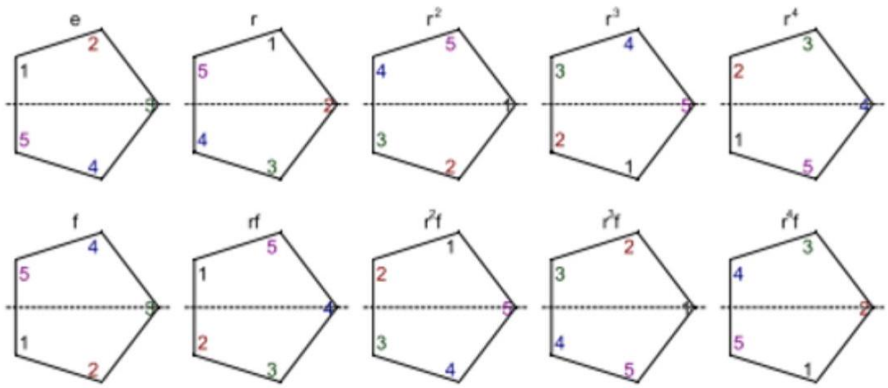


Table 2. Calay's table for $\boldsymbol{D_5}$

$\times$	e	r	r^2	r^3	r^4	$\boldsymbol{f}$	rf	r^2f	r^3f	r^4f
e	e	r	r^2	r^3	r^4	$\boldsymbol{f}$	rf	r^2f	r^3f	r^4f
r	r	r^2	r^3	r^4	e	rf	r^2f	r^3f	r^4f	$\boldsymbol{f}$
r^2	r^2	r^3	r^4	e	r	r^2f	r^3f	r^4f	$\boldsymbol{f}$	rf
r^3	r^3	r^4	e	r	r^2	r^3f	r^4f	f	rf	r^2f
r^4	r^4	e	r	r^2	r^3	r^4f	$\boldsymbol{f}$	rf	r^2f	r^3f
f	f	r^4f	r^3f	r^2f	rf	e	r^4	r^3	r^2	r
rf	rf	$\boldsymbol{f}$	r^4f	r^3f	r^2f	r	e	r^4	r^3	r^2
r^2f	r^2f	rf	$\boldsymbol{f}$	r^4f	r^3f	r^2	r	e	r^4	r^3
r^3f	r^3f	r^2f	rf	f	r^4f	r^3	r^2	r	e	r^2
r^4f	r^4f	r^3f	r^2f	rf	f	r^4	r^3	r^2	r	e

The elements of the group D_3 in cycle form is as follows.

$$D_5 = \left\{ \begin{array}{c} (1),(2,5)(3,4),(1,2)(3,5),(1,2,3,4,5),(1,3)(4,5),(1,3,5,2,4), \\ (1,4)(2,3),(1,4,2,5,3),(1,5,4,3,2),(1,5)(2,4) \end{array} \right\}$$

Routing calculations shows that the stabilizers of the points $1, 2, 3, 4$ and 5 are respectively given by:

$G_1 = \{(1),(2,5)(3,4)\}$

$G_2 = \{(1),(1,3)(4,5)\}$

$G_3 = \{(1),(1,5)(2,4)\}$

$G_4 = \{(1),(1,2)(3,5)\}$

$G_5 = \{(1),(1,4)(2,3)\}$

And the orbit of the points 1,2, and 3 is given by $1^G = 2^G = 3^G = 4^G = 5^G = \{1,2,3,4,5\}$

A group $G = D_5$ acting on a set $\Omega = \{1,2,3,4,5\}$ is said to be transitive on Ω if it has one orbit, and so $\alpha^{\mathrm{G}} = \Omega$ for all $\alpha \in \Omega$. A group which is not transitive is called intransitive. Thus,

$$D_5 = \left\{ \begin{array}{c} (1),(2,5)(3,4),(1,2)(3,5),(1,2,3,4,5),(1,3)(4,5),(1,3,5,2,4), \\ (1,4)(2,3),(1,4,2,5,3),(1,5,4,3,2),(1,5)(2,4) \end{array} \right\} \text{ is transitive.}$$

Also the stabilizer of each element is maximal in D3 as there exists no normal subgroup of order greater than 2. Thus, D5 is primitive.

3.2.3. GAP Results

We shall now construct dihedral groups of prime degrees and investigate their transitivity and primitivity using the group algorithm and programing (GAP).

GAP 4.12.2 built on 2022-12-19 10:30:03+0000

GAP

https://www.gap-system.org

Architecture: x86_64-pc-cygwin-default64-kv8

Configuration: gmp 6.2.1, GASMAN, readline

Loading the library and packages...

Packages: AClib 1.3.2, Alnuth 3.2.1, AtlasRep 2.1.6, AutPGrp 1.11, Browse 1.8.19, CaratInterface 2.3.4, CRISP 1.4.6, Cryst 4.1.25, CrystCat 1.1.10, CTblLib 1.3.4, curlInterface 2.3.1, FactInt 1.6.3, Forms 1.2.9, GAPDoc 1.6.6, genss 1.6.8, IO 4.8.0, IRREDSOL 1.4.4, LAGUNA 3.9.5, orb 4.9.0, Polenta 1.3.10, Polycyclic 2.16, PrimGrp 3.4.3, RadiRoot 2.9, recog 1.4.2, ResClasses 4.7.3, SmallGrp 1.5.1, Sophus 1.27, SpinSym 1.5.2, TomLib 1.2.9, TransGrp 3.6.3, utils 0.81 Try '??help' for help. See also '?copyright', '?cite' and '?authors'

```
gap>
gap> D3 := DihedralGroup(IsGroup 6);
Group([ (1,2,3), (2,3) ])
gap> Order (D3);
6
gap> Elements (D3);;
gap>
gap> IsAbelian (D3);
false
gap> IsTransitive (D3);
true
gap> IsPrimitive (D3);
true
```

```
gap>
gap> D5 := DihedralGroup(IsGroup, 10);
Group([ (1,2,3,4,5), (2,5)(3,4) ])
gap> Order (D5);
10
gap> Elements (D5);;
gap> IsAbelian (D5);
false
gap> IsTransitive (D5);
true
gap> IsPrimitive (D5);
true
gap> D7 := DihedralGroup(IsGroup, 14);
Group([ (1,2,3,4,5,6,7), (2,7)(3,6)(4,5) ])
gap> Order (D7);
14
gap> Elements (D7);;
gap> IsAbelian (D7);
false
gap> IsTransitive (D7);
true
gap> IsPrimitive (D7);
true
gap>
gap> D11 := DihedralGroup(IsGroup, 22);
Group([ (1,2,3,4,5,6,7,8,9,10,11), (2,11)(3,10)(4,9)(5,8)(6,7) ])
gap> Order (D11);
22
gap> Elements (D11);;
gap> IsAbelian (D11);
false
gap> IsTransitive (D11);
true
gap> IsPrimitive (D11);
true
gap>
gap> D13 := DihedralGroup(IsGroup, 26);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13), (2,13)(3,12)(4,11)(5,10)(6,9)(7,8) ])
gap> Order (D13);
26
gap> Elements (D13);;
gap> IsAbelian (D13);
false
gap> IsTransitive (D13);
true
gap> IsPrimitive (D13);
```

```
true
gap>
gap> D17 := DihedralGroup (IsGroup, 34);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17),
(2,17)(3,16)(4,15) (5,14)(6,13)(7,12) (8,11) (9,10) ])
gap> Order (D17);
34
gap> Elements (D17);;
gap> IsAbelian (D17);
false
gap> IsTransitive (D17);
true
gap> IsPrimitive (D17);
true
gap>
gap> D19 := DihedralGroup(IsGroup, 38);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19),
(2,19)(3,18)(4,17)(5,16)(6,15)(7,14) (8,13) (9,12)(10,11) ])
gap> Order (D19);
38
gap> Elements (D19);;
gap> IsAbelian (D19);
false
gap> IsTransitive (D19);
true
gap> IsPrimitive (D19);
true
gap>
gap> D23 := DihedralGroup(IsGroup, 46);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),
(2,23)(3,22)(4,21)(5,20) (6,19)(7,18)(8,17)(9,16)(10,15)(11,14)(12,13) ])
gap> Order (D23);
46
gap> Elements (D23);;
gap> IsAbelian (D23);
false
gap> IsTransitive (D23);
true
gap> IsPrimitive (D23);
true
gap>
gap> D27 := DihedralGroup(IsGroup, 54);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27),
(2,27)(3,26)(4,25)(5,24)(6,23)(7,22)(8,21)(9,20)(10,19)(11,18)(12,17)(13,16)(14,15) ])
gap> Order (D27);
54
```

```
gap> Elements (D27);;
gap> IsAbelian (D27);
false
gap> IsTransitive (D27);
true
gap> IsPrimitive (D27);
false
gap>
gap> D29 := DihedralGroup(IsGroup, 58);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29),
(2,29)(3,28)(4,27)(5,26)(6,25)(7,24)(8,23)(9,22)(10,21)(11,20)(12,19)(13,18)(14,17)(15,16) ])
gap> Order (D29);
58
gap> Elements (D29);;
gap> IsAbelian (D29);
false
gap> IsTransitive (D29);
true
gap> IsPrimitive (D29);
true
gap>
gap> D31 := DihedralGroup(IsGroup, 62);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,
26,27,28,29,30,31)(2,31)(3,30)(4,29)(5,28)(6,27)(7,26)(8,25)(9,24)(10,23)
(11,22)(12,21)(13,20)(14,19)(15,18)(16,17) ])
gap> Order (D31);
62
gap> Elements (D31);;
gap> IsAbelian (D31);
false
gap> IsTransitive (D31);
true
gap> IsPrimitive (D31);
true
gap>
```

Based on the trend in 3.2.1, 3.2.2 and 3.2.3 we proved a proposition which concerns particularly on transitivity and primitivity of all the dihedral groups of prime degree which are not p -group. This is the content of the next proposition and therefore it forms an important part of this work.

Proposition 3.1. *Let G be a dihedral group of degree p, where p is an odd prime number. Then G is (i) transitive and (ii) imprimitive.*

Proof. (i) That G is transitive follows easily from Lemma 2.6. Next, name the vertices of G as $1, 2, 3, \ldots,$ p and let l be the line of symmetry joining the vertex 1 and the middle of the vertices $\frac{p+1}{2}$ and $\frac{p+3}{2}$ so that

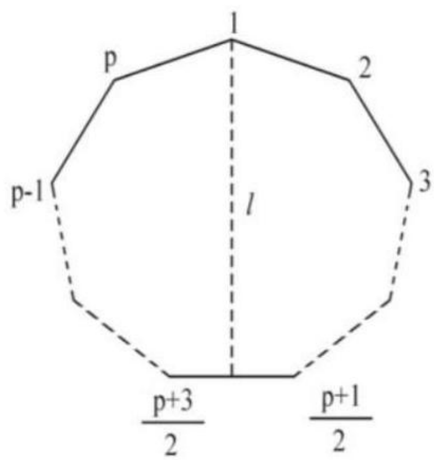
$$\alpha = (2,p)(3,p-1)(4,p-2)\ldots\ldots\left(\frac{p+1}{2},\frac{p+3}{2}\right) \tag{5}$$

is the reflection in l (see figure 5). Then $G_1 = \{(1), \alpha\}$ is the stabilizer of the point 1 . We readily see that G_1 is a non-identity proper subgroup of G which has

$$H = \left\{(1),(2,p),(3,p-1),(4,p-2),\ldots,\left(\frac{p+1}{2},\frac{p+3}{2}\right),\alpha\right\} \tag{6}$$

as a subgroup properly lying between G_1 and G, that is, $G_1 < H < G$. It follows by virtue of Theorem 2.1 that G is imprimitive.

Figure 5. Diagram for Dihedral Groups of Degree p.



□

4. Conclusion

The purpose of this research was to carry out further study on transitive and primitive dihedral groups of certain degrees. In particular, the ultimate goal was to determine the transitive and primitive nature of dihedral groups of prime degrees that are not p-groups using numerical approach. This entails generating dihedral groups of these degrees, studying, investigating and analyzing them so as to determine their transitivity and primitivity properties.

To do this, we set out specific objectives which were achieved as follows:

(i) All dihedral groups of prime degrees that are not p -groups were shown to be transitive and primitive.

(ii) The results in (i) above were validated using illustrations and a standard program namely Groups, Algorithms and Programming (GAP) version 4.11.1 of 2021.

4.1. Recommendations

We highly recommend that future research should further examine the groups been considered in this work to determine their nilpotency and regularity using numerical approach. This will further enhance already done works towards completion of the rewriting of the proofs of the Classification of the Finite Simple Groups (CFSG) that has been on course for a while now.

References

[1] Apine E, Jelten BN. Trends in transitive p-groups and their defining relations. J Math Theor Model 2014; 4(11): 192–209.

[2] Apine E, Jelten BN, Homti EN. Transitive 5-groups of degree $5^2 = 25$. Res J Math Stat 2015; 7(2): 17–19.

[3] Araújo JP, Cameron PJ, Hulpke A, Popes P. Imprimitive permutations in primitive groups. arXiv:1611.06450v1 [math.GR], 2016.

[4] Ben OJ, Adagba TT, Auta TJ. Analysis on properties and structure of dihedral groups. Afr J Math Stat 2024; 7(2): 51–68.

[5] Ben OJ, Hamma S, Adamu MS. On the transitivity and primitivity of permutation groups of degree $4p$ constructed via wreath products using numerical approach. Int J Math Anal Model 2022; 5(2): 254–263.

[6] Burness TC, Tong-Viet HP. Primitive permutation groups and derangements of prime power order. Manuscr Math 2016; 150(3): 255–291.

[7] Cameron PJ. Notes on finite group theory. Bull Lond Math Soc 2013; 13(1): 1–22.

[8] Cayley A. On the theory of groups as depending on the symbolic equation $\theta^n = 1$. Philos Mag 1854; 7(42): 40–47.

[9] Dixon JD, Mortimer B. Permutation Groups. Grad Texts Math, Vol. 163. New York, NY, USA: Springer, 1996.

[10] Fawcett JM. The base size of a primitive diagonal group. J Algebra 2013; 375(1): 302–321.

[11] Gallian JA. Contemporary Abstract Algebra. 7th ed. Brooks/Cole, Cengage Learning, 2010.

[12] Galois E, Singh AR. The last mathematical testament of Galois. 1897. Available from: https://www.ias.ac.in/article/fulltext/reso/004/10/0093-0100. (Retrieved August 2018).

[13] Gandi TI, Hamma S. Investigating simple and regular dihedral groups of an even degree regular polygon using the concept of p-groups. Frontiers Knowl Int J Pure Appl Sci 2018; 1.

[14] Gandi TI, Hamma S. Investigating solvable and nilpotent concepts on dihedral groups of an even degree regular polygon. Frontiers Knowl Int J Pure Appl Sci 2019; 2.

[15] Hulpke A, Bettima I, Martin S, Heiko T, Robert A. GAP – Groups, Algorithms and Programming, Version 4.8.5. Aachen–St Andrews, 2016.

[16] Kleiner I. The evolution of group theory: a brief survey. Math Mag 1986; 59(4): 195–202.

[17] Neerajah A, Subramanian P. Vertex graceful and difference labeling of some special graphs. Asia Mathematika 2025; 9(2).

[18] Roman S. Fundamentals of Group Theory: An Advanced Approach. New York, NY, USA: Birkhäuser, Springer, 2012.

[19] Passman DS. Permutation Groups. Math Lecture Notes Ser. New York, NY, USA: W.A. Benjamin, 1968: 255–279.

[20] Rotman JJ. An Introduction to the Theory of Groups. 4th ed. Springer-Verlag, 1979.