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A study on algebraic properties of permutation group using numerical approach

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Abstract: In this study, we carried out further study on permutation group of degrees n , where n is a positive integer. Commutativity and Transitivity are two pivotal properties that provide deeper insights into group structures. Commutativity in Group Theory refers to the property where the order of elements in a group operation does not affect the result. A group ($G, *$) is said to be commutative or abelian if: $a * b = b * a, \forall a, b \in G$. Transitivity, on the other hand, reflects a group's ability to act uniformly on a set, highlighting its symmetrical properties. A group G acting on a set Ω is said to be transitive on Ω if it has one orbit and so $\alpha^G = \Omega$ for all $\alpha \in \Omega$. Equivalently, G is transitive if for every pair of point $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$. In this work we generated some symmetric groups of degree n as a good example of permutation group and used computational tools, including Groups, Algorithms, and Programming (GAP) to analyze their structures and action properties and discuss their commutativity and transitivity. It was found that symmetric groups of degrees $n < 3$ are commutative and transitive while non commutative but transitive otherwise. These findings contribute to a deeper understanding of finite permutation groups, offering new insights into their classification and properties. This study not only enriches the theoretical framework of abstract algebra but also provides practical applications in areas such as cryptography and computational group theory.

Key words: commutative groups, transitive groups, permutation group, p-groups, numerical approach, group theory, GAP

1. Introduction

1.1. Background of the Study

Group theory is a fundamental area in abstract algebra, essential for studying mathematical structures and symmetry in various systems. It provides the framework to explore sets equipped with operations that satisfy particular axioms, including closure, associativity, identity, and invertibility.

Until about 1850, according to Cameron (2013) [7], the term 'group' referred to a set G of transformations of a set Ω, such that G closed under composition of functions, contains the identity transformation and the inverse of each of its elements. This implies that the function is one-to-one and onto, that is, a permutation. Any permutation group is an algebraic structure whose elements are all the possible permutations of a given set equipped with the binary operation of function composition ((Gallian, 2010 [9]) and (Roman, 2012 [15])).

According to Khukhro and Mazurov (2014) [12] and Müller, P. M. (2013) [14], several survey articles in the research space written about the implications of the classification of finite simple group for permutation groups reveal that finite permutation groups have been generated for research purposes using various approaches.

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Determination of such properties as commutativity, simplicity, transitivity, primitivity, solubility, almost-simplicity, nilpotency and regularity of various categories of groups have been carried out with the view to classify finite permutation groups by various authors (Li and Praeger, 2012 [13]). Apine and Jelten (2014) [1] achieved a classification of transitive and faithful p-groups (Abelian and Non-abelian) of degrees at most p^3 whose centre is elementary Abelian of rank two. Apine et al., (2015) [2] determined, up to equivalence, the actual transitive p-groups (Abelian and Non-abelian) of degree p^2 for $p = 5$ and achieved a classification of transitive 5 groups of degree 5^2. Ben et al, (2024) [4] on "Analysis of Properties and Structure of Dihedral Groups" formulated some new results and validated their claims using computational group theory.

Joseph-Louis Lagrange during his time observed permutation as arrangements, that is, as a list $i_1, i_2, \ldots, i_n$ with no repetition of any of the elements of Ω. The implication is, an arrangement, $i_1, i_2, \ldots, i_n$, define a function $\alpha : \Omega \to \Omega$ by $\alpha(j) = i$ for all $j \in \Omega$. Thus, every rearrangement gives a bijection (Burness and Tong-Viet, 2016 [6]). Let Ω be a nonempty set, a permutation of Ω is a bijection $\alpha : \Omega \to \Omega$. We denote the set of all permutations of Ω by S_Ω. When Ω is finite, that is, $\Omega = \{1, 2, \ldots, n\}$, we write S_n (the symmetric group of degree n) instead of S_Ω where $|S_n| = n!$ is the number of elements in S_n referred to as the order of the group S_n. Symmetric groups contain all possible permutations of a set of elements, while alternating groups contain only even permutations, allowing us to understand structural distinctions and behaviours within these groups.

Permutation groups are crucial for studying symmetries and transformations. Permutation groups, particularly symmetric and alternating groups, play a significant role in multiple areas, including cryptography, chemistry, and physics.

The algebraic properties of permutation groups hold importance due to their applications in various mathematical and practical domains. However, understanding these properties is challenging due to the intricate structures involved. This study aims to commutativity and transitivity of symmetric groups as good examples of permutation groups to enhance deeper understanding of permutation group theory.

Graphical and numerical methods, combined with computational tools such as GAP, offer effective techniques for studying these algebraic structures (Hulpke et al., 2016 and Johnson et al.[4, 6]).

2. Materials and Method

2.1. Introduction

In this work, knowledge of the basic facts from both the theory of abstract finite groups and the theory of permutation will be assumed throughout. Relevant theorems and results are quoted with example where necessary, in order to enhance proper understanding of the subject matter. We also use the Groups Algorithm and Programming (GAP) to enhance and validate our work.

2.2. Definition of Permutations and Permutation Group

A permutation is a bijective function mapping a set onto itself, arranging its elements in different orders. A permutation group is a group whose elements are permutations, with the group operation being function composition. Permutations are often represented in cycle notation, a concise way to denote element rearrangements.

2.3. Basic Theorems on Commutativity and Transitivity

2.3.1. Symmetric Group, S_n

The symmetric group S_n is the set of all permutations on n elements, equipped with composition as the operation. Symmetric groups are fundamental to group theory, serving as examples of nonAbelian groups when n > 2 and illustrating properties such as closure, identity, and invertibility. The order of the symmetric group of degree n (where $n \in \mathbf{N}$) is $n!$ as proved by the following theorem.

Theorem 2.1 (Cameron, 2013 [7]). $|S_n| = n!$.

Proof. $|S_n|$ is just the number of ways the integers 1 through n can be arranged. In other words, in how many different ways can we fill the blanks?

$$\begin{pmatrix} 1 & 2\dots & n \\ - & -\dots & - \end{pmatrix}$$

Well, we have n choices for the first entry and then $n-1$ choices for the next entry, and so on yielding a total of $n \cdot (n-1)\dots 1 = n!$ Total choices.
For example $|S_3| = 3! = 6$ just as $|S_2| = 2! = 2$. □

2.3.2. Alternating Group, A_n

The alternating group A_n is the subset of S_n containing only even permutations. A_n is simple for n ≥ 5 and plays a crucial role in finite simple groups. Even permutations are compositions of an even number of transpositions, distinguishing A_n from S_n. The order of the alternating group of degree n, $|A_n| = |S_n|/2$.

2.3.3. Algebraic Properties of Permutation Groups

Permutation groups satisfy the algebraic properties of closure, associativity, identity, and inverses. These properties form the foundation of group operations within S_n and A_n, making these groups ideal for studying algebraic structures and transformations.

2.3.4. Subgroups and Normal Subgroups in Permutation Groups

Subgroups of permutation groups include all subsets that are themselves groups under the same operation. Normal subgroups, invariant under conjugation, play a significant role in the structural analysis of S_n and A_n, impacting properties like solvability and simplicity.

Theorem 2.2 (Cayley, 1854 [8]). *Any finite group G is isomorphic to a subgroup of the symmetric group S_n of degree n, where $n = |G|$.*

Proof. Let G act on itself by right multiplication $g^h = gh$ for all $g, h \in G$. If $g^h = g$ then $gh = g$ and so $h = 1$. That is, the kernel of the action is $\{1\}$. The mapping $f : G \to \mathrm{sym}(G)$ define by $f : g \to f_g$ where $\alpha f_g = \alpha^g$ for any $\alpha \in G$ is a homomorphism. Then $G/\ker f \cong \mathrm{im}\, f$. But ker $f = \{1\}$ and $\mathrm{im}\, f \leq \mathrm{sym}(G) = S_n$. Accordingly $G \leq S_n$. In general we have that if G acts on Ω with k kernel of the action then $G/k \leq \mathrm{sym}(\Omega)$. □

2.3.5. The Structure of S_n (Bäärnhielm, 2014 [3])

As always, S_{n} is the group of bijections or permutations of a set of n objects, say $\mathrm{X}=\{1,2,\ldots,\mathrm{n}\}$. Its group operation is the composition of bijections. We will frequently refer to the objects being permuted as letters. Recall the notation $\sigma(12\cdots n)=\sigma(1)\sigma(2)\cdots\sigma(n)$. To simplify the notation, we will denote σ by writing $\sigma=[\sigma(1),\sigma(2),\ldots,\sigma(n)]$. We will also frequently denote the identity element of S_{n} by (1).

Let α and β be two elements in $\mathrm{S_n}$ such that

$$\alpha=\begin{pmatrix}1&2&3&4&\ldots&n\\3&2&1&4&\ldots&n\end{pmatrix}\text{ and }\beta=\begin{pmatrix}1&2&3&4&\ldots&n\\2&1&3&4&\ldots&n\end{pmatrix}.$$

$$\text{Then, }\beta\alpha=\begin{pmatrix}1&2&3&4&\ldots&n\\3&1&2&4&\ldots&n\end{pmatrix}\text{ and }\alpha\beta=\begin{pmatrix}1&2&3&4&\ldots&n\\2&3&1&4&\ldots&n\end{pmatrix}.$$

Theorem 2.3. *Any k-cycle in S_{n} can be written as a product of transpositions (two cycles). (Here $n>1$ or else we have $S_1=\{e\}$).*

Proof. If we have a 1 -cycle, then it is the identity element which can be written as $(1,2)^2=(1,2)(1,2)=e$. Now if we have ak-cycle were $k\geq 2$ then we can work out the product just as $(a_1,a_2,\ldots,a_k)=(a_1,a_2)(a_1a_3)\ldots(a_1a_k)$.

□

2.3.6. Orbit (Müller, 2013 [14])

When a group G acts on a set Ω, a typical point α is moved by elements of G to various other elements in the set Ω. The set of these images is called the orbit of α under G, and we denote it by $\alpha^G:=\{\alpha^x\mid x\in G\}$. Thus, Ω is a union of disjoint orbits, say $\Omega=\cup_{i=1}^{s}\Omega_{\mathrm{i}}$. A group G acting on a set Ω is said to be transitive on Ω if it has one orbit and so $\alpha^G=\Omega$ for all $\alpha\in\Omega$. Equivalently, G is transitive if for every pair of point $\alpha,\beta\in\Omega$ there exists $g\in G$ such that $\alpha^g=\beta$. A group which is not transitive is called intransitive.

2.3.7. Commutativity

A group G is commutative if for all $x,y\in G,xy=yx$.

3. Results and Discussion

3.1. Introduction

Throughout this chapter, unless otherwise explicitly indicated, "n" is a positive integer.

3.2. Commutativity and Transitivity of Symmetric Groups of Degree n.

The following are the main results on the constructed symmetric groups of degree n.

3.2.1. The Symmetric Group of Degree $\mathbf{n}(\mathbf{n}=\mathbf{1})$

The symmetric group G of degree 1 is a permutation group with one element namely the identity element. That is $\{\mathrm{e}\}$.

Thus, $G=S_1=\{\mathrm{e}\}$ as all the axioms of a group are trivially satisfied.
Since G is a one-element group, it is commutative as well as transitive.

3.2.2. The Symmetric Group of Degree $\mathbf{n(n=2)}$

The symmetric group G of degree 2 is a permutation group with two elements namely the identity element and one other element say "a". That is $\{\mathrm{e},\mathrm{a}\}$.

Thus, $G = S_2 = \{\mathrm{e},\mathrm{a}\}$.

We write $S_2 = \{\mathrm{e},\mathrm{a}\}$, where $e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $\mathrm{a} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

In cycle form, $S_2 = \{(1),(1,2)\}$.

The order of $S_2 = 2$, denoted by $|S_2| = 2$.

Table 1. Cayley's table for $G = \{(1),(1,2)\}$

$\circ$	(1)	$(1,2)$
(1)	(1)	$(1,2)$
$(1,2)$	$(1,2)$	(1)

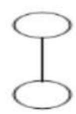

Figure 1. Cayley's diagram for $G = \{(1),(1,2)\}$

(1) and S_2 are the only normal subgroups of S_2 since S_2 is commutative.

The orbit of the points 1 and 2 in S_2 are given by $1^G = 2^G = \{1,2\}$ implying that

$$\alpha^{\mathrm{G}} = \Omega \text{ for all } \alpha \in \Omega.$$

Thus, S_2 is transitive.

3.2.3. The Symmetric Group of Degree $\mathbf{n(n=3)}$

The symmetric group G of degree 3 is a permutation group with $3! = 6$ elements namely,

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Thus, $G = S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$.

In cycle form, $S_3 = \{(1),(2,3),(1,2),(1,3),(1,2,3),(1,3,2)\}$.

The order of $S_3 = 6$, denoted by $|S_3| = 6$.

$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\} = \{(1),(1,2,3),(1,3,2)\}$ forms a group called the alternating group of degree $3, A_3$, which is the only proper normal subgroup of S_3.

Clearly, $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}^{\circ} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^{\circ} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Hence S_3 is not commutative.

The orbit of the points 1,2 and 3 in S_3 are given by $1^G = 2^G = 3^G = \{1,2,3\}$ implying that

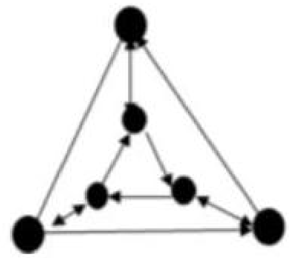

Figure 2. Cayley's diagram for $G = S_3$

$$\alpha^{\mathrm{G}} = \Omega \text{ for all } \alpha \in \Omega.$$

Thus, S_3 is transitive.

3.2.4. The Symmetric Group of Degree $\mathbf{n (n = 4)}$

The symmetric group G of degree 4 is a permutation group with $4! = 24$ elements namely

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

In cycle form, $S_4 = \{(1), (12), (13), (14), (34), (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243), (243), (1234), (1324), (1342), (1423), (1432)\}$.

The order of $S_4 = 4! = 24$, denoted by $|S_4| = 24$.

$\{(1), (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243)\ \}$ forms a group called the alternating group of degree $4, A_4$ which is the only proper normal subgroup of S_4.

$(12)(13) = (132) \neq (123) = (13)(12)$. Hence, S_4 is not commutative.

The orbit of the points 1,2 and 3 in S_3 are given by $1^G = 2^G = 3^G = 4^G = \{1, 2, 3, 4\}$ implying that

$$\alpha^{\mathrm{G}} = \Omega \text{ for all } \alpha \in \Omega.$$

Thus, S_4 is transitive.

3.2.5. GAP Results

We shall now construct symmetric groups of degrees n and investigate their commutativity and transitivity using the group, algorithm and programing GAP 4.12.2. [10]

```
[breaklines=true]
```

```
GAP 4.12.2 built on 2022-12-19 10:30:03+0000
GAP https://www.gap-system.org
Architecture: x86_64-pc-cygwin-default64-kv8
Configuration: gmp 6.2.1, GASMAN, readline
Loading the library and packages ...
Packages: AClib 1.3.2, Alnuth 3.2.1, AtlasRep 2.1.6, AutPGrp 1.11, Browse 1.8.19,
Try '??help' for help. See also '?copyright', '?cite' and '?authors'
gap>
gap> S1 := SymmetricGroup (1);
Group(())
gap> Order(S1);
1
gap> Elements(S1);
[()]
gap> Orbit(S1,1);
[1]
gap> IsAbelian(S1);
true
gap> IsTransitive(S1);
true
gap>
gap> S2 := SymmetricGroup(2);
Sym([ 1 .. 2 ])
gap> Order(S2);
2
gap> Elements(S2);

[breaklines=true]
[ (), (1,2) ]
gap> Orbit(S2,1);
[ 1, 2 ]
gap> Orbit(S2,2);
[ 1, 2 ]
gap> IsAbelian(S2);
true
gap> IsTransitive(S2);
true
gap>
gap> S3:= SymmetricGroup(3);
Sym([ 1 .. 3 ])
gap> Order(S3);
6
gap> Elements(S3);
[(), (2,3), (1,2), (1,2,3), (1,3,2), (1,3)]
gap> Orbit(S3,1);
[ 1, 3, 2]
```

```
gap> Orbit(S3,2);
[ 1, 3, 2 ]
gap> Orbit(S3,3);
[ 1, 3, 2 ]
gap> IsAbelian(S3);
false
gap> IsTransitive(S3);
true
gap>
gap> S4 := SymmetricGroup(4);
Sym([ 1 ..4 ])
gap> Order(S4);
24
gap> Elements(S4);
[(), (3,4), (2,3), (2,3,4), (2,4,3), (2,4), (1,2), (1,2)(3,4), (1,2,3), (1,2,3,4), (1,2,4,3), (1,2,4),
(1,3,2), (1,3,4,2), (1,3), (1,3,4), (1,3)(2,4), (1,3,2,4), (1,4,3,2), (1,4,2), (1,4,3), (1,4),
(1,4,2,3), (1,4)(2,3)]
gap> Orbit(S4,1);
[ 1, 4, 2, 3]
gap> Orbit(S4,2);
[ 1, 4, 2, 3]
gap> Orbit(S4,3);
[ 1, 4, 2, 3]
gap> Orbit(S4,4);
[ 1, 4, 2, 3]
gap> IsAbelian(S4);
false
gap> IsTransitive(S4);
true
```

```
[fontsize=\small,breaklines=true]
gap>
gap> S5 := SymmetricGroup(5);
Sym([ 1 .. 5])
gap> Order(S5);
120
gap> Elements(S5);
```

[(), (4,5), (3,4), (3,4,5), (3,5,4), (3,5), (2,3), (2,3)(4,5), (2,3,4), (2,3,4,5), (2,3,5,4), (2,3,5), (2,4,3), (2,4,5,3), (2,4), (2,4,5), (2,4)(3,5), (2,4,3,5), (2,5,4,3), (2,5,3), (2,5,4), (2,5), (2,5,3,4), (2,5)(3,4), (1,2), (1,2)(4,5), (1,2)(3,4), (1,2)(3,4,5), (1,2)(3,5,4), (1,2)(3,5), (1,2,3), (1,2,3)(4,5), (1,2,3,4), (1,2,3,4,5), (1,2,3,5,4), (1,2,3,5), (1,2,4,3), (1,2,4,5,3), (1,2,4), (1,2,4,5), (1,2,4)(3,5), (1,2,4,3,5), (1,2,5,4,3), (1,2,5,3), (1,2,5,4), (1,2,5), (1,2,5,3,4), (1,2,5)(3,4), (1,3,2), (1,3,2)(4,5), (1,3,4,2), (1,3,4,5,2), (1,3,5,4,2), (1,3,5,2), (1,3), (1,3)(4,5), (1,3,4), (1,3,4,5), (1,3,5,4), (1,3,5), (1,3)(2,4), (1,3)(2,4,5), (1,3,2,4), (1,3,2,4,5), (1,3,5,2,4), (1,3,5)(2,4), (1,3)(2,5,4), (1,3)(2,5), (1,3,2,5,4), (1,3,2,5), (1,3,4)(2,5), (1,3,4,2,5), (1,4,3,2), (1,4,5,3,2), (1,4,2), (1,4,5,2), (1,4,2)(3,5), (1,4,3,5,2), (1,4,3), (1,4,5,3), (1,4), (1,4,5), (1,4)(3,5), (1,4,3,5), (1,4,2,3), (1,4,5,2,3), (1,4)(2,3), (1,4,5)(2,3), (1,4)(2,3,5), (1,4,2,3,5), (1,4,2,5,3), (1,4,3)(2,5), (1,4)(2,5,3), (1,4,3,2,5), (1,4)(2,5), (1,4,2,5), (1,5,4,3,2), (1,5,3,2), (1,5,4,2), (1,5,2), (1,5,3,4,2), (1,5,2)(3,4), (1,5,4,3), (1,5,3), (1,5,4), (1,5), (1,5,3,4), (1,5)(3,4), (1,5,4,2,3), (1,5,2,3), (1,5,4)(2,3), (1,5)(2,3), (1,5,2,3,4), (1,5)(2,3,4), (1,5,3)(2,4), (1,5,2,4,3), (1,5,3,2,4), (1,5)(2,4,3), (1,5,2,4), (1,5)(2,4)]

```
gap> Orbit(S5,1);
[ 1, 5, 2, 3, 4 ]
gap> Orbit(S5,2);
[ 1, 5, 2, 3, 4 ]
gap> Orbit(S5,3);
[ 1, 5, 2, 3, 4 ]
gap> Orbit(S5,4);
[ 1, 5, 2, 3, 4 ]
gap> Orbit(S5,5);
[ 1, 5, 2, 3, 4]
gap> IsAbelian(S5);
false
gap> IsTransitive(S5);
true
gap> S6 := SymmetricGroup(6);
Sym([ 1 .. 6 ])
gap> Order(S6);
720
gap> Elements(S6);
```

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[ 1, 6, 2, 3, 4, 5]
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gap> IsAbelian(S6);
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gap> IsTransitive(S6);
true
gap>
```

Based on the trend in 4.2.1, 4.2.2, 4.2.3, 4.2.4, and 4.2.5, we proved a proposition that concerns particularly on the commutativity and transitivity of symmetric groups of degree $n \geq 1$. This forms an important part of this work.

3.3. Proposition

Let G be a symmetric group of degree n, where n is a positive integer. Then G is (i) commutative and transitive if $\mathrm{n} \leq 2$ and (ii) non-commutative but transitive if $\mathrm{n} > 2$.

Proof. (i) Case $\mathrm{n} = 1$

Let G be a symmetric group of degree $1, (S_1)$. Then G contains only the identity element e, since every group must have an identity element. It follows that $G = \{\mathrm{e}\}$ satisfies all the axioms of a group and is commutative and transitive since it is a singleton set.

Case $\mathrm{n} = 2$
Now let G be a symmetric group of degree 2, (S_2). Then $G = \{\mathrm{e}, \mathrm{a}\} \cong \mathrm{Z}_2$. With the mapping $\mathrm{e} \to 0$ and a $\to 1$, the Cayley's table for G is as follows.

Table 2. Cayley's table for $G = \{\mathrm{e}, \mathrm{a}\}$

$\circ$	e	a
e	e	a
a	a	e

Clearly, $G = \{\mathrm{e}, \mathrm{a}\}$ is commutative since it has only two elements and (a)•(a)=e. Also, S_2 acts transitively on $\mathrm{X} = \{1, 2\}$ because for any two elements x, y, there exists a permutation (specifically (12)) that maps x to y. Thus, any symmetric group G, of degree n , where $\mathrm{n} \leq 2$ is commutative (abelian) and transitive.

(ii) Case $n \geq 3$
Case S_3
The symmetric group S_3 consists of the permutations:

$$S_3 = \{\mathrm{e}, (12), (13), (23), (123), (132)\} \text{ where:}$$

e is the identity, (12), (13), (23) are transpositions (swap two elements), while (123) and (132) are cyclic permutations.

It is obvious some elements do not commute. For instance:

$$(12) \cdot (13) = (132) \neq (123) = (13) \cdot (12).$$

Thus, S_3 is not commutative since not all elements commute.
Also, S_3 acts transitively on $\mathrm{X} = \{1, 2, 3\}$ since for every pair of point $x, y \in X$ there exists $g \in G$ such that $x^g = y$.
S_3 is not commutative but transitive.
In general, let G be a symmetric group of degree $\mathrm{n} > 2$. Then $G = S_\mathrm{n}$ is the group of bijections or permutations of a set of n objects, say $\mathrm{X} = \{1, 2, \ldots, \mathrm{n}\}$. Its group operation is the composition of bijections. We will frequently refer to the objects being permuted as letters. Recall the notation $\sigma(12 \cdots n) = \sigma(1)\sigma(2) \cdots \sigma(n)$. To simplify the notation, we will denote σ by writing $\sigma = [\sigma(1),\ \sigma(2), \ldots, \sigma(\mathrm{n})]$. We will also frequently denote the identity element of S_n by (1). Clearly G is transitive since for any two elements x, y, there is some permutation in S_n that maps x to y .
Now let α and β be two elements in G such that

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 3 & 2 & 1 & 4 & \dots & n \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 1 & 3 & 4 & \dots & n \end{pmatrix}.$$

Then, $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & \\ 3 & 1 & 2 & 4 & \dots & n \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 3 & 1 & 4 & \dots & n \end{pmatrix} = \alpha\beta$. Thus, G is non-commutative.

Thus, any symmetric group G, of degree n, where $n \geq 3$ is non-commutative (abelian) but transitive. □

3.4. Illustrating Examples

3.4.1. Example 1: Symmetric Group of Degree n(n = 5)

The Symmetric Group of Degree $n(n = 5)$ is denoted S_5 $|S_5| = n! = 120$.

$S_5 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}, \dots\dots\dots, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\}$. These can be represented in cycle form as follows.

```
{(1), (4,5), (3,4), (3,4,5), (3,5,4), (3,5), (2,3), (2,3)(4,5), (2,3,4), (2,3,4,5), (2,3,5,4),
(2,3,5), (2,4,3), (2,4,5,3), (2,4), (2,4,5), (2,4)(3,5), (2,4,3,5), (2,5,4,3), (2,5,3),
(2,5,4), (2,5), (2,5,3,4), (2,5)(3,4), (1,2), (1,2)(4,5), (1,2)(3,4), (1,2)(3,4,5), (1,2)(3,5,4),
(1,2)(3,5), (1,2,3), (1,2,3)(4,5), (1,2,3,4), (1,2,3,4,5), (1,2,3,5,4), (1,2,3,5), (1,2,4,3),
(1,2,4,5,3), (1,2,4), (1,2,4,5), (1,2,4)(3,5), (1,2,4,3,5), (1,2,5,4,3), (1,2,5,3), (1,2,5,4),
(1,2,5), (1,2,5,3,4), (1,2,5)(3,4), (1,3,2), (1,3,2)(4,5), (1,3,4,2), (1,3,4,5,2), (1,3,5,4,2),
(1,3,5,2), (1,3), (1,3)(4,5), (1,3,4), (1,3,4,5), (1,3,5,4), (1,3,5), (1,3)(2,4), (1,3)(2,4,5),
(1,3,2,4), (1,3,2,4,5), (1,3,5,2,4), (1,3,5)(2,4), (1,3)(2,5,4), (1,3)(2,5), (1,3,2,5,4), (1,3,2,5),
(1,3,4)(2,5), (1,3,4,2,5), (1,4,3,2), (1,4,5,3,2), (1,4,2), (1,4,5,2), (1,4,2)(3,5), (1,4,3,5,2),
(1,4,3), (1,4,5,3), (1,4), (1,4,5), (1,4)(3,5), (1,4,3,5), (1,4,2,3), (1,4,5,2,3), (1,4)(2,3),
(1,4,5)(2,3), (1,4)(2,3,5), (1,4,2,3,5), (1,4,2,5,3), (1,4,3)(2,5), (1,4)(2,5,3), (1,4,3,2,5),
(1,4)(2,5), (1,4,2,5), (1,5,4,3,2), (1,5,3,2), (1,5,4,2), (1,5,2), (1,5,3,4,2), (1,5,2)(3,4),
(1,5,4,3), (1,5,3), (1,5,4), (1,5), (1,5,3,4), (1,5)(3,4), (1,5,4,2,3), (1,5,2,3), (1,5,4)(2,3),
(1,5)(2,3), (1,5,2,3,4), (1,5)(2,3,4), (1,5,3)(2,4), (1,5,2,4,3), (1,5,3,2,4), (1,5)(2,4,3),
(1,5,2,4), (1,5)(2,4) }
```

Now let α and β be two elements in $G = S_5$ such that

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}.$$

Then, $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} = \alpha\beta$. Thus, G is non-commutative.

The orbit of the points $1, 2, 3, 4, 5$ is given by $1^G = 2^G = 3^G = 4^G = 5^G = \{1, 2, 3, 4, 5\}$
Thus, S_5 is transitive.

4. Conclusion

The purpose of this research was to carry out further study on commutative and transitive permutation groups of any degree. In particular, the ultimate goal was to determine the commutative and transitive nature of symmetric groups of any degree. This entails generating symmetric groups of various degrees, studying, investigating and analyzing them so as to determine their commutativity and transitivity.

To do this, we set out specific objectives which were achieved as follows:
i) All symmetric groups of degrees $n \leq 2$ are commutative and transitive while those of degrees 3 and above are non-commutative but transitive.
ii) The results in (i) above were validated using illustrations and a standard program namely Groups, Algorithms and Programming (GAP) version 4.12.2 of 2022.

4.1. Recommendations

We highly recommend that future research should further examine the groups been considered in this work to determine their nilpotency and regularity using numerical approach. This will further enhance already done works towards completion of the rewriting of the proofs of the Classification of the Finite Simple Groups (CFSG) that has been on course for a while now.

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